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UNIVERSITY OF ALBERTA

GENERALIZED HEISENBERG ALGEBRAS AND THEIR VERTEX OPERATOR REPRESENTATIONS

by

MARC A FABBRI

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULLFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

SPRING, 1992

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ISBN 0-315-73095-1

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DEGREE FOR WHICH THESIS WAS PRESENTED: DOCTOR OF PHILOSOPHY

YEAR THIS DEGREE WAS GRANTED: 1992

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Dedicated

To my Father and Mother

with love and appreciation

ABSTRACT

Kac-Moody algebras have proven to be a far-reaching generalization of the finite-dimensional Lie algebras of Killing and Cartan. These objects arise from the group of polynomial maps from the circle into a Lie group. Their representation theory and that of the closely related Virasoro algebra is of interest to both physicists and mathematicians.

More recently, physicists have been trying to understand the effects of developing physical theories modeled over the Lie algebra corresponding to the group of polynomial maps from a torus into a Lie group - the so-called torus algebra. Mathematicians, including Moody, Eswara Rao and Yokonuma, have studied the torus algebra in its fullest sense and they name it the *toroidal algebra*. Moreover, they have used the vertex operators from dual resonance theories in physics to initiate a representation theory of the algebra and, in doing so, have produced a completely decomposable module for it.

In my work, I propose a new Lie algebra called the *Virasoro-toroidal algebra* which contains the Virasoro algebra as a subalgebra and the toroidal algebra as an ideal. From the point of view of conformal field theory or vertex operator algebras, it is a more natural algebra to consider than the toroidal algebra alone. Due to the enormous size of the toroidal algebras, it is helpful to identify certain key subalgebras. This is done by introducing a class of Lie algebras called *Generalized Heisenberg algebras*. With the use of vertex operators and certain oscillator representations of the Virasoro algebra, I study the corresponding effect on the representation theory. The end result is that complete decomposability over the toroidal algebra transforms essentially into complete reducibility over the Virasoro-toroidal algebra.

ACKNOWLEDGEMENTS

I am most grateful to my supervisor R.V. Moody for both his efforts in teaching and for his guidance during the development of my thesis.

I would like to express my appreciation to Bruce Allison, Steven Berman and Arturo Pianzola for many stimulating lectures during the ongoing Edmonton-Saskatoon seminar. Additional thanks goes to John Coleman, Hans Kummer, Keqin Liu and Frank Okoh for their interest in my work and their instructive comments.

Louise Gratton and her wonderful family has been most supportive and helpful during my time in Edmonton and to them I am thankful.

Finally I would like to point out that it has been with my close friend Robb Fry (a unique and thoroughly unforgetable character) that I have experienced the joys and sorrows of becoming a mathematician.

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PHYSICAL ORIGINS: SEMI-DIRECT PRODUCTS IN NATURE

It is well-known that the representation theory of Kac-Moody Lie algebras is of interest to both physicists and mathematicians. These algebras arise from considering the Lie algebra of polynomial maps from the unit circle S^1 into a compact Lie group G, ie. the loop group [PS].

More recently, there has been some activity in the direction of producing physical field theories by considering the algebra of the group of polynomial maps from the torus $T^1 = S^1 \times S^1$ into a compact Lie group G ([Ba], [Bo] and [F]). Moreover, mathematicians have investigated the "torus algebra", or rather, various central extensions of it ([MY], [Y] and [MEY]). This algebra exists in its fullest sense in [MEY], where it is called the *toroidal Lie algebra* and an attractive vertex operator representation theory is introduced.

Motivated by the work in [MEY], I propose here a new Lie algebra called the *Virasoro-toroidal Lie algebra* which contains the toroidal algebra as an ideal and the Virasoro algebra as a subalgebra. The purpose of this lengthy introduction is to give both physical motivation for suggesting the algebra and mathematical confirmation that it is the more natural one by outlining how its representation theory compares with that of the toroidal algebra.

PART A: The Harmonic Oscillator Algebra

Many important groups and algebras in mathematics and physics occur naturally as a semi-direct product of two other groups or algebras. A few examples are the dihedral groups, the group of Euclidean motions, the space groups of crystallography and solid state physics and the inhomogeneous Poincaré group of special relativity (see Mackey [Ma]).

Another instance of this kind of structure is the harmonic oscillator algebra. Our goal here is to outline the physical origin of this object and indicate how its representations arise in the physical theory. This object is the finite dimensional analogue of some of the objects we will work with, and its canonical representation is typical of some representations that we will study.

Let $\mathfrak{a}$ be a three dimensional complex vector space with basis $\langle x,y,z\rangle$. That is, $\mathfrak{a} := \mathbf{C}x \oplus \mathbf{C}y \oplus \mathbf{C}z$. Introduce a multiplication $[\cdot,\cdot]$ on $\mathfrak{a}$ as follows. Let $[\cdot,\cdot] : \mathfrak{a} \times \mathfrak{a} \longrightarrow \mathfrak{a}$ be the unique skew-symmetric ($[a,b]=-[b,a]$, $a,b\in\mathfrak{a}$) bilinear map satisfying

$$(1) \qquad \begin{cases} [x,y] = z \\ [x,z] = 0 = [y,z] \end{cases}$$

We say that the element z is central since it commutes with every element in $\mathfrak{a}$. In fact, $\mathbf{C}z$ is the entire centre and by (1) the product of any two elements in $\mathfrak{a}$ belongs to $\mathbf{C}z$. A consequence of this is that the Jacobi identity, $[[a,b],c] + [[b,c],a] + [[c,a],b] = 0$, $a,b,c \in \mathfrak{a}$ is trivially satisfied. We call $\mathfrak{a}$ the *three-dimensional Heisenberg Lie algebra*.

A look into the origin of the name given to $\mathfrak{a}$ takes one to the heart of what is unquestionably one of the greatest intellectual achievements in the twentieth century - Quantum mechanics. In one sentence, quantum mechanics is a physical theory designed to describe the behaviour of objects at the atomic scale. That is, quantum mechanics is the study of small scale structure.

It was a number of discoveries in the field of thermodynamics late in the nineteenth century that got the ball rolling. Interestingly enough, this is roughly the same point in history when *Wilhelm Killing* was led to classify the simple finite dimensional Lie algebras over the complex numbers. In 1900, *Max Planck* formulated his theory of heat radiation which culminates in the famous formula $E = h\nu$ where $h = 6.7 \times 10^{-27}$ cm·gr·s is called Planck's constant. His theory suggested that radiant energy was emitted in the form of discrete packets - an idea that was revolutionary in its consequences but one that was not easily accepted. It wasn't until 1918 that he was awarded the Nobel prize in physics for this work. Confirmation of Planck's hypothesis came when *Albert Einstein* used it to give an explanation of the *photoelectric effect*.

Although some anomalies were removed from physics, still others remained. In 1923 *Compton* showed experimentally that waves (high frequency light) behaved like particles and *DeBroglie* suggested that particles also behave like waves and he gave an expression for the wavelength λ in terms of Planck's constant h and the momentum p of the particle: $\lambda = h/p$. This was not verified experimentally until 1926, by Davisson and Germer, after the completion of quantum mechanics.

In 1913 *N. Bohr* had introduced the concept of stationary orbits and combined this with Planck's hypothesis to explain the hydrogen spectrum. In 1925 *Heisenberg, Born* and *Jordan* introduced matrix mechanics. This arose out of an initial attempt by Heisenberg to remove unobservable orbits from the theory of Bohr and to replace them with the observable differences in orbital radii. The table of differences $r_i - r_j$ formed a matrix r and similarly the table of momentum differences $p_i - p_j$ formed a matrix p and in the paper by the three men [HBJ] the essence of quantum mechanics, namely $[r,p] = i\hbar$, was distilled.

In 1925 *Schroedinger* presented his wave mechanics which was an alternate way of obtaining the results of matrix mechanics. Within a few months Schroedinger managed to show that the two theories were equivalent ways of representing the commutation relations.

One striking consequence of these mathematical theories was that they indicated certain limitations in quantum mechanics which did not exist in classical physics. The most famous example of this is the *Heisenberg Uncertainty Principle*. We will see that the three-dimensional Heisenberg algebra $\mathfrak{a}$ above is a mathematical expression of what the Uncertainty Principle says.

In its first formulation the Uncertainty Principle stated that, in a quantum dynamical system, it is impossible to measure simultaneously and with arbitrary accuracy both the position and the momentum of a particle.

Let us imagine an isolated electron. Motivated by the classical theory of light where the intensity of light is measured by the square of the amplitude of a wave, in quantum mechanics one uses a wave function Ψ and

measures, via $|\Psi(x)|$, the probability of finding the electron at the point x. We will be satisfied with knowing that the electron is localized to some interval Δx. This then requires a wave function which is zero outside Δx and is nonzero inside. Clearly such a wave cannot be made up of a single wavelength. In fact, the wavelengths must be chosen so that the corresponding waves cancel each other outside Δx and enhance each other inside Δx. In fact, arbitrarily short wavelengths λ are required.

Now certainly when the λ's decrease, the uncertainty Δx in measuring x decreases. But De'Broglie's equation for matter waves states that $p\lambda=\hbar$, where $\hbar$ is Planck's constant, and so Δp increases when λ decreases. Combining, we have that Δx decreases when Δp increases and conversely.

This can all be put together quantitatively to obtain the celebrated inequality:

$$\Delta x \, \Delta p \geq \hbar \tag{2}$$

This suggests that a measurement in one observable A (such as position) may effect a measurement in another observable B (such as momentum). We call two observables *incompatible* if this is the case. Otherwise they are said to be compatible.

Now in quantum mechanics, one associates a Hermitian operator $\hat{A}$ to each observable A in a natural way and it is possible to prove that under this correspondence we have

$$\text{A and B are compatible} \iff [\hat{A},\hat{B}] = 0 \tag{3}$$

where $[\hat{A},\hat{B}]$ denotes the commutator of $\hat{A}$ and $\hat{B}$ (see [S]). For example, in what is called the *position representation*, one associates the operator of left multiplication by x, on the polynomial ring $\mathbb{C}[x]$, to the observable *position*

and the operator $-i\hbar \frac{d}{dx}$ to the observable *momentum*. Note that $[-i\hbar \frac{d}{dx}, x] = -i\hbar \neq 0$ and hence by (3) position and momentum are incompatible observables which is consistent with what we saw in (2). Moreover, we have:

Proposition A: The unique linear map $\phi : \mathfrak{a} \longrightarrow \mathfrak{gl}(\mathbb{C}[x])$ satisfying

$$\begin{cases} x \longmapsto x \\ y \longmapsto -i\hbar \frac{d}{dx} \\ z \longmapsto i\hbar \end{cases}$$

is an irreducible representation of $\mathfrak{a}$ on $\mathbb{C}[x]$.

For $\kappa \in \mathbb{C}$ let $d_\kappa : \mathfrak{a} \longrightarrow \mathfrak{a}$ be the unique derivation satisfying $d_\kappa(x) = \kappa x$, $d_\kappa(y) = -\kappa y$ and $d_\kappa(z) = 0$. Make the one-dimensional space $\mathbb{C}d_\kappa$ into an abelian Lie algebra and let $\tilde{\mathfrak{a}}_\kappa := \mathbb{C}d_\kappa \oplus \mathfrak{a}$, as a $\mathbb{C}$-space. Then $\tilde{\mathfrak{a}}_\kappa$ becomes a Lie algebra in a unique way such that $\mathbb{C}d_\kappa$ and $\mathfrak{a}$ are subalgebras and $[d_\kappa, a] = d_\kappa(a)$, $a \in \mathfrak{a}$. We call $\tilde{\mathfrak{a}}_\kappa$ the *semi-direct product* of $\mathbb{C}d_\kappa$ and $\mathfrak{a}$ and we write $\tilde{\mathfrak{a}}_\kappa = \mathbb{C}d_\kappa \propto \mathfrak{a}$. The semi-direct product symbol $\propto$ should not be confused with the Greek letter α. The idea of a semi-direct product and its relationship to derivations occurs in E. Cartan's thesis, 1894, on p.112-113 where it is used to show that all derivations of a semi-simple Lie algebra are inner (often called Zassenhaus' theorem). Our next goal is to show that the algebra $\tilde{\mathfrak{a}}_\kappa$ arises naturally in the context of the harmonic oscillator problem.

A *harmonic oscillator* is a particle P attracted to a fixed point O by a force proportional to the distance from O to P.

Let x denote the distance from O to P. By Newton's equation of motion we have: $-\alpha x = F = ma = m \frac{d^2x}{dt^2}$ where α is the constant of proportionality

called *Hooke's constant*. Let $\omega^2 := \frac{\alpha}{m}$. ω is called the *angular frequency* of the oscillator. Then the equation of motion is

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

The solution to this differential equation is given by $x(t) = \cos(\omega t)$ and so one can see why this system is called a harmonic oscillator.

Now to solve the harmonic oscillator problem amounts to determining the admissible energy levels when the system is quantized. That is, to determine the energy levels by calculating the eigenvalues of the Hermitian operator which corresponds to the observable energy. These eigenvalues will all be real numbers since the operator is Hermitian. For a classical oscillator any positive energy value can be obtained, but as we will see this is not the case for the corresponding quantum system.

Classically the total energy of the system is given by the sum of the kinetic and potential energies: $H = K + V$

where $K := \frac{1}{2} mv^2 = \frac{1}{2} \frac{p^2}{m}$ since momentum $p=mv$ and V is defined by the equation $\frac{dV}{dx} = -F = \alpha x = (\omega^2 m)x$. Thus $V = \frac{1}{2} \omega^2 m x^2$ and hence

(4) $$H = \frac{1}{2} \frac{p^2}{m} + \frac{1}{2} \omega^2 m x^2.$$

Now in (4) we could replace x by left multiplication on $\mathbb{C}[x]$ and p by differentiation $-i\hbar\frac{d}{dx}$ and then H would be an operator on $\mathbb{C}[x]$ whose eigenvalues could be calculated. However, the genius of Dirac provided

quantum mechanics with a formalism which, for the purpose of finding the eigenvalues of H, allows us to avoid this position representation and all other specific representations. Roughly speaking, one assumes that the roles of the observables x and p are played by a pair of Hermitian operators and that these operators satisfy the axiom $[x,p] = i\hbar$. Note that this is precisely the relation that we obtained earlier in the position representation. Whatever these operators are, when we replace x and p by them in (4) we have a Hermitian operator for H and we now indicate Dirac's representation-free method of extracting its eigenvalues. Introduce the *raising and lowering operators*

$$a^+ := \frac{1}{\sqrt{2\hbar\omega m}}\,(p + i\omega m x)$$

$$a^- := \frac{1}{\sqrt{2\hbar\omega m}}\,(p - i\omega m x)$$

Since $[x,p] = i\hbar$ and $H = \hbar\omega\left(a^+a^- + \frac{1}{2}\right) = \hbar\omega\left(N + \frac{1}{2}\right)$

where $N := a^+a^-$ we have

(5)
$$\begin{cases} [a^-,a^+] = I \\ [H,a^\pm] = \pm\hbar\omega a^\pm \end{cases}$$

Now let w be an eigenvector of H with eigenvalue λ, ie. $Hw = \lambda w$. Then $Ha^\pm(w) = (a^\pm H \pm \hbar\omega a^\pm)(w) = (\lambda \pm \hbar\omega)a^\pm(w)$ and so $a^\pm(w)$ is an eigenvector of H with eigenvalue $\lambda \pm \omega\hbar$. Since H is a nonnegative Hermitian operator its

eigenvalues are real and nonnegative. Let λ_0 be the smallest eigenvalue of H with corresponding eigenvector w_0. Then $a^-(w_0) = 0$. This gives

$$0 = \hbar\omega\, a^+a^-(w_0) = \hbar\omega N(w_0) = \left(H - \frac{1}{2}\hbar\omega\right)(w_0) \quad \text{and so} \quad Hw_0 = \frac{\hbar\omega}{2}(w_0).$$

In summary we have a sequence of distinct eigenvalues

$$\lambda_0 = \frac{1}{2}\hbar\omega\ ,\ \lambda_1 = \frac{3}{2}\hbar\omega\ ,\ldots,\ \lambda_n = \left(n + \frac{1}{2}\right)\hbar\omega\ ,\ldots$$

with corresponding linearly independent eigenvectors

$$\hat{v}_0 = w_0\ ,\quad \hat{v}_1 = a^+(w_0),\ldots,\ \hat{v}_n = (a^+)^n(w_0),\ldots\ .$$

Since eigenvectors corresponding to distinct eigenvalues of a Hermitian operator are orthogonal, normalizing the $\hat{v}_i$ gives us an orthonormal basis which we denote by $\{v_1, v_2, \ldots\}$.

Now let $V := \coprod_{n\geq 0} \mathbb{C}v_i$ and let $\tilde{\alpha} := \tilde{\alpha}_{\hbar\omega}$ (That is, take $\kappa=\hbar\omega$).

Proposition B: The unique linear map $\tilde{\phi} : \tilde{\alpha} \longrightarrow \mathfrak{gl}(V)$ satisfying

$$\begin{cases} x \longmapsto a^- \\ y \longmapsto a^+ \\ z \longmapsto I \\ d \longmapsto H \end{cases}$$

is an irreducible representation of $\tilde{\alpha}$ on V.

Next we seek to find the explicit matrix representation of the linear maps a^-, a^+ and H. First define c_n, $n>0$ by $v_{n+1} := c_{n+1}\, a^+(v_n)$. Then

$$
\begin{aligned}
1 = \langle v_{n+1} | v_{n+1} \rangle &= c_{n+1}\overline{c_{n+1}} \langle a^+ v_n | a^+ v_n \rangle \\
&= |c_{n+1}|^2 \langle v_n | a^- a^+ v_n \rangle \\
&= |c_{n+1}|^2 \langle v_n | (N+1) v_n \rangle \\
&= |c_{n+1}|^2 \langle v_n | (\frac{H}{\hbar\omega} + \frac{1}{2})(v_n) \rangle \\
&= |c_{n+1}|^2 \left(\frac{1}{2} \langle v_n | v_n \rangle + (n + \frac{1}{2}) \langle v_n | v_n \rangle \right) \\
&= |c_{n+1}|^2 (n+1)
\end{aligned}
$$

and hence $|c_{n+1}| = (n+1)^{-1/2}$. We may choose $c_{n+1} = (n+1)^{-1/2} \in \mathbb{R}$.

Now $\delta_{k,n+1} = \langle v_k | v_{n+1} \rangle = \langle v_k | c_{n+1} a^+ (v_n) \rangle = c_{n+1} \langle v_k | a^+ (v_n) \rangle$

which gives $(a^+)_{kn} = (n+1)^{1/2} \delta_{k,n+1}$.

Similarly $(a^-)_{kn} = (n+1)^{1/2} \delta_{k+1,n}$. Thus we can write

$$
a^+ = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots \\
\sqrt{1} & 0 & 0 & 0 & \cdots \\
0 & \sqrt{2} & 0 & 0 & \cdots \\
0 & 0 & \sqrt{3} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{pmatrix}
\quad \text{and} \quad
a^- = \begin{pmatrix}
0 & \sqrt{1} & 0 & 0 & \cdots \\
0 & 0 & \sqrt{2} & 0 & \cdots \\
0 & 0 & 0 & \sqrt{3} & \cdots \\
0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{pmatrix}.
$$

There is an important mathematical distinction to be made regarding the representations of $\mathfrak{a}$ on $\mathbb{C}[x]$ and $\tilde{\mathfrak{a}}$ on V appearing in propositions A and B. The latter representation is an example of what is called a *highest weight representation* whereas the former is not. The property that V possesses which $\mathbb{C}[x]$ does not is what one calls a *weight space decomposition*. If we write $\tilde{\mathfrak{h}} := \mathbb{C}z \oplus \mathbb{C}d_\kappa \subset \tilde{\mathfrak{a}}$ and for $\alpha \in \tilde{\mathfrak{h}}^*$ we define the *weight space of* V *with respect*

$$V^\alpha := \{v \in V : h \cdot v = \alpha(h)v, \ \forall h \in \tilde{\mathfrak{h}}\}$$

then V decomposes into one-dimensional weight spaces $V^{\lambda_n} = \mathbb{C}v_n$, $n \geq 0$ where

$\lambda_n \in \tilde{\mathfrak{h}}^*$ is defined by $\lambda_n := n\lambda$ where $\lambda(c) := 0$ and $\lambda(d_\kappa) := n$. That is, $V = \coprod_{n\geq 0} V^{\lambda_n}$. Such a decomposition is not possible for $\mathbb{C}[x]$ with respect to $\mathfrak{h} := \mathbb{C}c$ since the subalgebra $\mathfrak{h}$ cannot produce the necessary "separation". $\tilde{\mathfrak{a}}$ is an example of what is called a *Lie algebra with triangular decomposition* (see [MP-2]) and V is an example of what is called a *Verma module*. Since such objects have a well-developed theory, we consider V to be the nicer module.

PART B: Vertex Operators in physics

Let us return to the fascinating history of physics. Since Rutherford's discovery of the satellite-description of the atom, more than 200 different elementary particles have been discovered. Physicists have been trying, in various ways, to classify these into families. One method is according to their *spin*. Those with half-integer spins, like $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, such as electrons, protons and neutrons, are called *fermions* and those with whole-integer spins, like 1, 2, 3,..., such as photons, are called *bosons*. One relevance of this classification is that the bosons are responsible for the fundamental forces of nature. They are the "carriers" of the weak and strong nuclear forces, the electromagnetic force and (it is hypothesized) the gravitational force.

Another method of classification is according to the strength with which particles interact. For example, not all particles experience the strong nuclear force. The *leptons*, which include the electrons and the neutrinos interact only using the weak nuclear force and the electromagnetic force. In contrast, the *hadrons*, which include the proton and neutron, interact using the strong nuclear force and other forces.

Following the example of *Rutherford*, in an attempt to understand the inner secrets of an elementary particle, physicists use particle accelerators to shoot electrons and α-particles with extremely high speeds at the elementary particles. Such experiments are called *scattering experiments* and can be viewed as black-box-type experiments in the sense that one tries to relate what goes in to what comes out.

To keep track of the experimental results, in 1937, *John Wheeler* introduced his famous *S-matrix*. The matrix was to contain all significant information that can be discovered about the particle. One only needs to interpret the information correctly. A significant step in speeding up this process was taken by *Richard Feynman*, Wheeler's student. Feynman showed that the perturbation theory for the S-matrix could be put into one-to-one correspondence with certain diagrams. He, in turn, showed that the diagrams could be classified, for example, according to the number of loops that they contain. The result clarified the theory of *quantum electrodynamics*. In the event of electromagnetic interactions, Feynman's approach gave a practical means for computing the S-matrix. The graphical technique developed by Feynman was a pictorial representation of a perturbation expansion of the S-matrix in powers of the dimensionless coupling constant $\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$. The series is almost certainly not convergent and probably asymptotic. No proofs exist. Also in the evaluation of the higher order terms (loops) in the series, divergent integrals occur. These divergences can be eliminated from the theory by replacing the parameters (mass and charge) that occur in the theory by their "computed" values. This process is called *Renormalization*. Renormalizarion depends crucially on the fact that the coupling constant α is dimensionless.

For hadronic (strong) interactions this was not possible until the advent of gauge theories because the coupling constant without gauge couplings had dimensions.

Although the Feynman method did not work in the case of hadron-hadron interactions, all was not lost. An Italian physicist by the name of Tulio Regge was studying mathematical properties of the S-matrix associated to what are called *resonances*. Resonances are short-lived phenomena that occur during the scattering experiment. The outcome of a collision between two high energy particles was, in some cases, the creation of new particle and, in other cases, a temporary localization of energy. There was often debate as to whether these resonances were actually new particles or simply a "ringing of energy". What was particularly notable was that these resonances where *extended* in space and that they possessed a very large spin value, uncharacteristic of the hadrons.

While other physicists unsuccessfully tried to discover a set of rules to distinguish the various resonances, Regge was concerned purely with the mathematical description of the S-matrix. He studied the following properties of the S-matrix, for simple non-relativistic quantum scattering. Consider

$$S(E,\theta) = \sum S(E,\ell)\ P_\ell(\cos\theta)$$

where $P_\ell(\cos\theta)$ is a Legendre polynomial. Now consider extending the discrete variable ℓ in $S(E,\theta)$ to a complex variable and study the behavior of the function $S(E,\ell)$ as an analytic function of ℓ. Regge found that poles in $S(E,\ell)$ for $\mathrm{Im}\ell$ not too large correspond to resonances in the scattering amplitude. This idea was taken up by particle physicists (never by Regge himself) to develop the so-called *Regge trajectories* or *Regge plots*.

The next major contribution came, in 1968, from Gabriele Veneziano who found a single model relating the various resonances, collisions and the scattering processes and Regge trajectories. His method is called the *dual resonance model* [Ven].

Veneziano thought of the S-matrix as a black box in which there were two sets of input. These are called the s and t-scattering channels. In short, his idea was that the two channels give "dual" descriptions of the same physics. He claimed that what appeared to be a resonance or ringing of energy in one channel would look like an interaction in the other. In the former case, the two incoming particles would collide and join to form a new collective and temporary mode called a resonance **R** and in the latter case they would interact via the exchange of a force particle τ (see diagram 1 below).

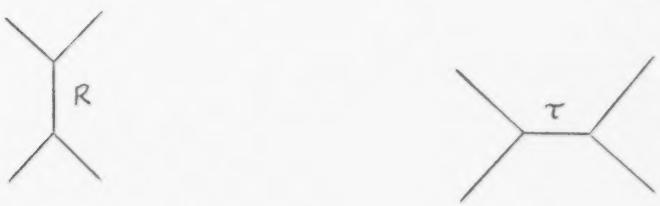


Diagram 1.

Physicists puzzled over how two completely different physical processes could produce the same outcome. Although Veneziano could not give a physical explanation of his model, he had developed a formula which incorporated duality. The model was of limited validity for computation of experimental results.

Veneziano's model captured the attention and interest of many physicists yet there was one more major shortcoming. It allowed for the existence of *ghosts*. Ghosts are what physicists define as states of negative norm. These can be thought of as events which occur with negative probability. It turns out that one way of avoiding ghosts is to reformulate the theory in a 26-dimensional space. If one is willing to work in these multidimensional spaces then the dual resonance theory seems to produce harmony in the world of elementary particles.

Motivated by the observation that elementary particles seem to behave like extended objects in space, in 1970 Yoichiro Nambu proposed the idea that elementary particles are not points or quarks but rather rotating and vibrating strings in space. Notice that when elementary particles are viewed in this way then the two figures in diagram 1 become topologically equivalent:

Diagram 2.

Modern string theory as expounded, for example, in [M] describes a string as a 2-dimensional sheet moving in a d-dimensional space (d=26 in the Veneziano model). In the mathematical formalism of this theory, one quantizes strings and the position/momentum variables are replaced by Hermitian

operators on a Hilbert space (as in part A). This leads to an infinite dimensional analogue of the 3-dimensional Heisenberg algebra arising from the harmonic oscillator discussed above. This new algebra can be described abstractly as the Lie algebra with generators x_k, $k\in\mathbb{Z}$ and c with relations

$$(6)\qquad \begin{cases} [x_k, x_\ell] := k\delta_{k+\ell,0}c \\ c \text{ is central.} \end{cases}$$

Now in this theory the conditions which are imposed to prevent the appearance of ghosts are called the *subsidiary conditions* [Vir]. They correspond to the vanishing of the energy-momentum components, whose Fourier modes are the famous *Virasoro operators*.

For $k\in\mathbb{Z}$ these are defined by $L_k := \frac{1}{2}\sum_{j\in\mathbb{Z}} :x_{-j}x_{j+k}:$ where the colon denotes what is called *normal ordering* and the x_ℓ are the generators of the infinite Heisenberg algebra in (6). L_0 is the Hamiltonian for the system and hence corresponds to the operator H in (4). Using the commutation relations in (6) one can show

$$(7)\qquad [L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3-m)\,\delta_{m+n,0}\mathfrak{z}$$

where $\mathfrak{z}$ is a scalar operator. The Lie algebra with generators L_k, $k\in\mathbb{Z}$ and $\mathfrak{z}$ with relations (7) and $\mathfrak{z}$ central is called the *Virasoro algebra* and is denoted *Vir*.

As in all physical theories, one needs to develop an operator formalism which deals first with free systems and then with systems which have

interactions. The *Vertex Operators* were invented to have a mathematical way of describing string propagation. One requirement for their construction was that they "transform" nicely under the Virasoro algebra. The reason for this is that it gives one a method of constructing new physical states from old ones.

Although the Vertex operators $X(\alpha,z)$, $z\in\mathbb{C}$ and $\alpha\in L$, where L is an even integral lattice, are quite complicated to define (we introduce them in ch.4) we point out that if one expands $X(\alpha,z)$ in a formal power series,

$$X(\alpha,z) = \sum_{n\in\mathbb{Z}} X_n(\alpha)z^{-n},$$

then the *moments*, $X_n(\alpha)$, are transformed under the Virasoro algebra as follows:

(8) $$[L_k, X_m(\alpha)] = \left\{ \frac{k}{2}(\alpha|\alpha) - (m+k) \right\} X_{m+k}(\alpha).$$

The number $J := \frac{(\alpha|\alpha)}{2}$ is called the *conformal dimension* of $X_m(\alpha)$.

Now let us consider the algebra $\mathcal{A}$ generated by the operators $X_m(\alpha)$, $m\in\mathbb{Z}$ and $\alpha\in L$ under operator commutation. Then (8) is a statement expressing the fact that together, $\mathcal{A}$ and $\mathcal{V}ir$, form a larger algebra which is a semi-direct product of these two algebras in such a way that $\mathcal{V}ir$ is a subalgebra and $\mathcal{A}$ is an ideal. Since the very construction of the Vertex operators is done so that they will transform under $\mathcal{V}ir$, if one is to extract an algebra from this physical situation, it is reasonable to expect the $X_m(\alpha)$ and the L_k should be taken together. This fact is already apparent in the development of the representation theory of affine Kac-Moody algebras.

Let us recall that the affine Kac-Moody algebra can be realized as the

$\mathbb{C}$-space $\mathfrak{g} := \dot{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c$ where $\dot{\mathfrak{g}}$ is a simple finite dimensional Lie algebra over $\mathbb{C}$ and $\mathbb{C}[t,t^{-1}]$ is the Laurent polynomial ring and c is some symbol. $\mathfrak{g}$ has multiplication given for $x,y \in \dot{\mathfrak{g}}$ and $k,\ell \in \mathbb{Z}$ by

$$(9) \qquad \begin{cases} [x\otimes t^k,\ y\otimes t^\ell] := [x,y]\otimes t^{k+\ell} + (x|y)k\delta_{k+\ell,0}c \\ c \text{ is central} \end{cases}$$

where $(\cdot|\cdot)$ denotes the Killing form on $\dot{\mathfrak{g}}$.

Now in 1980, Frenkel and Kac [FK] showed that the affine Kac-Moody algebra $\mathfrak{g}$ (corresponding to a simple finite dimensional Lie algebra $\dot{\mathfrak{g}}$ of type A_ℓ, D_ℓ of E_ℓ) admits a representation via vertex operators $X_m(\alpha)$, $m\in\mathbb{Z}$ and $\alpha\in\dot{Q}$ where $\dot{Q}$ is the root lattice of $\dot{\mathfrak{g}}$. This is called the *Basic representation*. In the literature it has become customary to work, not with $\mathfrak{g}$, but with a one-dimensional extension of $\mathfrak{g}$. In fact, one works with the semi-direct product algebra $\tilde{\mathfrak{g}} := \mathbb{C}L_0 \propto \mathfrak{g}$. This was the point that we alluded to earlier.

It is interesting to note that if $\dot{\mathfrak{h}}$ is a fixed (abelian) Cartan subalgebra of $\dot{\mathfrak{g}} = \mathfrak{sl}_2(\mathbb{C})$ then the subalgebra of $\mathfrak{g}$ generated by the subspace $\dot{\mathfrak{h}}\otimes_{\mathbb{C}}\mathbb{C}[t,t^{-1}]$ has commutation relations which are precisely those in (6). It is a remarkable fact that the affine Kac-Moody algebras are a perfect abstract model for the commutation relations occuring in physical theories such as quantum field theory and the dual resonance theory.

PART C: Thesis Overview

Recently, Moody, Eswara and Yokomuna, [MEY], have introduced a class of infinite dimensional Lie algebras, which generalize the Kac-Moody algebras, called toroidal Lie algebras. For any positive integer $n \geq 1$, one can realize the toroidal Lie algebra, denoted $\tau_{[n]}$, as the universal central extension of the Lie algebra $\dot{g} \otimes_{\mathbb{C}} \mathbb{C}[t_1^{\pm},\ldots,t_n^{\pm}]$ where $\dot{g}$ is a simple finite dimensional Lie algebra and $\mathbb{C}[t_1^{\pm},\ldots,t_n^{\pm}]$ denotes the associative commutative algebra of Laurent polynomials in $t_1,\ldots,t_n$. Of course, when $n=1$ we recover the affine Kac-Moody algebras.

The main achievement in [MEY] was to produce vertex representations of $\tau_{[2]}$ and study the detailed structure of the representations obtained. In contrast to the basic representation of Frenkel and Kac (see part B of this introduction), where they show that the Kac-Moody algebra is represented using $X_m(\alpha)$, $m \in \mathbb{Z}$ and $\alpha \in \dot{Q}$, the toroidal Lie algebra is represented using the moments $X_m(\alpha)$ with $m \in \mathbb{Z}$ and $\alpha \in \Delta^{re} \subset Q$ where Δ^{re} denotes the real roots of the affine root system corresponding to $\dot{Q}$. In [MEY] the authors prove that a certain Fock space is completely decomposable over $\tau_{[2]}$. That is, it can be written as a direct sum of indecomposable $\tau_{[2]}$-modules. They point out the difficulty in producing irreducible representations. The objective of my thesis is to address this latter situation.

Motivated by the discussions in part B of this introduction, we propose here a new Lie algebra called the *Virasoro-toroidal Lie algebra* which contains the toroidal Lie algebra as an ideal and the Virasoro algebra as a subalgebra. That is, we introduce a semi-direct product Lie algebra

$\tilde{\tau}_{[2]} := \mathcal{V}ir \propto \tau_{[2]}$. In order to extend the representation of the toroidal algebra to the full Virasoro-toroidal algebra, we use certain oscillator representations of the Virasoro algebra.

To examine the overall effect of the representation theory, we proceed by investigating certain key subalgebras. Letting n=1 be our guide, we look in $\tau_{[2]}$, for what would be an analogue of the principal Heisenberg algebra in (6). We are led to a class of Lie algebras, called the *Generalized Heisenberg Algebras*, denoted $\mathcal{H}(L,n)$, and defined for a geometric lattice L and a positive integer n. Although, like the Heisenberg algebra in (6), $\mathcal{H}(L,n)$ has the property that its derived algebra is equal to its centre, the centre is now infinite dimensional. When L is a rank 1 positive definite lattice, $\mathcal{H}(L,n)$ can be described as the Lie algebra with generators x_r, $z_r(s)$, $r,s \in \mathbb{Z}^n$ and relations

$$(10) \qquad \begin{cases} [x_r, x_s] := z_r(r+s) \\ z_r(s) \text{ is central} \end{cases}$$

together with the following linear relations on the central elements $z_r(s)$:
$z_r(s) + z_{r'}(s) := z_{r+r'}(s)$, $\alpha z_s(r) := z_{\alpha s}(r)$ and $z_r(r) := 0$.

As in [MEY], we focus on the case n=2. It is important to note that we will frequently vary the lattice L in $\mathcal{H}(L,n)$ and use both degenerate and nondegenerate ones. Moreover, lattices of definite, semi-definite and indefinite type will be used. In particular, an independent oscillator representation theory for $\mathcal{V}ir$ is developed in order to handle certain

representations of our algebras.

In the end, we find that complete decomposability over $\tau_{[2]}$ transforms "essentially" into complete reducibility over $\tilde{\tau}_{[2]}$. We say essentially since our Fock space decomposes into submodules K(m), $m \in \mathbb{Z}$ and each K(m) is irreducible except one, namely K(0). We show that K(0) is reducible by producing a filtration of submodules of it.

GENERALIZED HEISENBERG ALGEBRAS AND THE CANONICAL REPRESENTATION

Introduction: At the beginning of the section 1 the reader is introduced to a class of Lie algebras called *Generalized Heisenberg Algebras.* For a given geometric lattice L and positive integer n, the algebra $\mathcal{H}(L,n)$ is defined by means of an abstract presentation. When n=1 and L is a positive definite rank 1 lattice we recover the well-known infinite Heisenberg algebra. When n=1 we will denote $\mathcal{H}(L,1)$ by $\mathfrak{a}(L)$.

In section 2 we consider the case where L is the root lattice $\dot{Q}$ of a simple finite dimensional Lie algebra $\dot{\mathfrak{g}}$ over $\mathbb{C}$, and we give a concrete realization of $\mathcal{H}(\dot{Q},n)$ as a certain subalgebra of the toroidal Lie algebra $\tau_{[n]}$ introduced in [MEY].

In section 3 we restrict our attention to the case n=2. Throughout this work we will arbitrarily fix $\dot{Q}$ of type A_ℓ, D_ℓ or E_ℓ and we denote $\mathcal{H}(\dot{Q},2)$ by $\mathcal{H}$. The purpose of this section is to isolate a subalgebra of $\mathcal{H}$ which, after suitable extension by derivations, admits a triangular decomposition. We will be focusing on this subalgebra when developing the representation theory of $\mathcal{H}$.

Finally, in section 4, we construct the canonical representation of $\mathfrak{a}(L)$ from which all representations that we consider arise, and in section 5 we identify the canonical representation by means of an induced module.

§ 1.1 The Generalized Heisenberg Algebras

Let $(L,(\cdot|\cdot))$ be a (geometric) lattice. That is, a free $\mathbb{Z}$-module L of finite rank together with a nontrivial symmetric $\mathbb{Z}$-bilinear form $(\cdot|\cdot) : L \times L \longrightarrow \mathbb{Z}$. Let $\ell := \mathbb{C}\otimes_{\mathbb{Z}} L$ and extend $(\cdot|\cdot)$ to a symmetric $\mathbb{C}$-bilinear form (also denoted $(\cdot|\cdot)$) on ℓ. We call the lattice L nondegenerate if $(\cdot|\cdot)$ is nondegenerate on ℓ. See appendix A for generalities on geometric lattices.

Fix $n\in\mathbb{Z}$ with $n\geq 1$ and for each $r\in\mathbb{Z}^n$, let $\ell(r)$ be an isomorphic copy of ℓ and let $\mathfrak{Z}'(r)$ be an isomorphic copy of $\mathbb{C}^n$. Thus, as γ (resp. s) runs over a basis of ℓ (resp. $\mathbb{C}^n$), $\gamma(r)$ (resp. $z_s(r)$) runs through a basis of $\ell(r)$ (resp. $\mathfrak{Z}'(r)$).

Consequently, for $r,s,s'\in\mathbb{Z}^n$ and $\alpha\in\mathbb{C}$ we have

$$\text{(1.1.1)}\qquad \begin{cases} z_s(r) + z_{s'}(r) = z_{s+s'}(r) \\ \alpha\, z_s(r) = z_{\alpha s}(r) \end{cases}$$

Next define $\mathfrak{Z}' := \left(\coprod_{r\in\mathbb{Z}^n} \mathfrak{Z}'(r)\right)$, $R := \left(\coprod_{r\in\mathbb{Z}^n} \mathbb{C}z_r(r)\right)$ and $\mathfrak{Z} := \mathfrak{Z}'/R$. Consider the $\mathbb{C}$-space

$$\text{(1.1.2)}\qquad \mathcal{H}(L,n) := \left(\coprod_{r\in\mathbb{Z}^n} \ell(r)\right) \oplus \mathfrak{Z}.$$

Introduce a multiplication, $[\cdot,\cdot]$, on $\mathcal{H}(L,n)$ by defining

$$\text{(1.1.3)}\qquad \begin{cases} [\gamma(r),\eta(s)] = (\gamma|\eta)z_{r+s} \\ [\mathcal{H}(L,n),\mathfrak{Z}] = 0 = [\mathfrak{Z},\ \mathcal{H}(L,n)]. \end{cases}$$

By (1.1.3), $\mathcal{H}(L,n)$ is a two-step nilpotent algebra and hence the multiplication satisfies the Jacobi identity. Moreover, combining (1.1.1) and (1.1.3), we see that the multiplication in $\mathcal{H}(L,n)$ is skew-symmetric and hence it is a Lie algebra called the *Generalized Heisenberg algebra* associated to L and n.

Proposition 1.1.1: **(i)** $\mathcal{H}(L,n)$ is generated by $\left(\coprod_{r\in\mathbb{Z}^n} \ell(r) \right)$ unless L is totally isotropic.

(ii) $\dim_{\mathbb{C}} \mathfrak{Z} = \begin{cases} 1 , & n=1 \\ \infty , & n\geq 2. \end{cases}$

(iii) $\text{centre}\left(\mathcal{H}(L,n)\right) = \ell(0) \oplus \mathfrak{Z} \oplus \left(\coprod_{\substack{r\in\mathbb{Z}^n\setminus\{0\} \\ \gamma \in \text{rad}(\cdot\,|\,\cdot)}} \mathbb{C}\gamma(r) \right)$.

(iv) If n=1 and L is nondegenerate then $\dim_{\mathbb{C}}\text{centre}\left(\mathcal{H}(L,n)\right) = \text{rank}(L)+1$. Otherwise $\dim_{\mathbb{C}}\text{centre}\left(\mathcal{H}(L,n)\right) = \infty$.

Proof: For (i) note that if $\gamma\in L$ is nonisotropic then for any $r,s \in \mathbb{Z}^n$ we have $[\gamma(r),\gamma(s-r)] = (\gamma\,|\,\gamma)z_r(s)$. Since the case $n\geq 2$ is clear, to prove (ii) we only consider the case n=1. Recall $\mathfrak{Z} = \mathfrak{Z}'/R$ where $\mathfrak{Z}' = \coprod_{r\in\mathbb{Z}} \mathfrak{Z}'(r)$ and for each $r\in\mathbb{Z}$, $\mathfrak{Z}'(r) = \left\{z_s(r) : s\in\mathbb{C}\right\}$ has $\mathbb{C}$-linear structure given by (1.1.1).

Thus for $r\neq 0$, $z_s(r) = (s/r)\, z_r(r)$ and hence $\mathfrak{Z}'(r) = \mathbb{C}z_r(r)$. If r=0 we have $z_s(0) = s\, z_1(0)$ and hence $\mathfrak{Z}'(0) = \mathbb{C}z_1(0)$.

Now $\mathfrak{Z}' = \left(\coprod_{r\in\mathbb{Z}\setminus\{0\}} \mathfrak{Z}'(r) \right) \oplus \mathfrak{Z}'(0) = \left(\coprod_{r\in\mathbb{Z}\setminus\{0\}} \mathbb{C}z_r(r) \right) \oplus \mathbb{C}z_1(0)$

$$= \left(\coprod_{r\in\mathbb{Z}} \mathbb{C}z_r(r) \right) \oplus \mathbb{C}z_1(0) = R \oplus \mathbb{C}z_1(0)$$

and hence $\mathfrak{z} = \mathfrak{z}'/R \cong \mathbb{C}z_1(0)$. This proves (ii). To prove (iii), note that by (1.1.3), $\text{centre}\big(\mathcal{H}(L,n)\big) \supset \ell(0) \oplus \mathfrak{z} \oplus \Big(\coprod_{\substack{r\in\mathbb{Z}^n\setminus\{0\} \\ \gamma \in \text{rad}\,(\cdot|\cdot)}} \mathbb{C}\gamma(r) \Big)$. For the reverse inclusion, suppose $x \in \text{centre}\big(\mathcal{H}(L,n)\big) \cap \Big(\coprod_{r\in\mathbb{Z}^n\setminus\{0\}} \ell(r) \Big)$. It suffices to show that $x \in \Big(\coprod_{\substack{r\in\mathbb{Z}^n\setminus\{0\} \\ \gamma \in \text{rad}\,(\cdot|\cdot)}} \mathbb{C}\gamma(r) \Big)$. Write $x = \sum_{i=1}^{m} \gamma_i(r_i)$ where $\gamma_i \in \ell$ and $r_i \in \mathbb{Z}^n\setminus\{0\}$ are distinct $1\le i\le m$. For any $s\in\mathbb{Z}^n\setminus\{0\}$ and $\eta\in\ell$, we have

$$0 = [x,\eta(s)] = \sum_{i=1}^{m} [\gamma_i(r_i),\eta(s)] = \sum_{i=1}^{m} (\gamma_i|\eta) z_{r_i}(r_i+s).$$

Since the r_i are distinct, so are the r_i+s. Thus $(\gamma_i|\eta)=0$, $\forall\eta\in\ell$ and $1\le i\le m$ by the linear independence of the $z_{r_i}(r_i+s)$. Therefore $\gamma_i \in \text{rad}(\cdot|\cdot)$, $1\le i\le m$.

Finally, (iv) follows immediately from (iii) and the fact if L is nondegenerate then $\Big(\coprod_{\substack{r\in\mathbb{Z}^n\setminus\{0\} \\ \gamma \in \text{rad}\,(\cdot|\cdot)}} \mathbb{C}\gamma(r) \Big) = \{0\}$. ∎

Corollary 1.1.2 $\mathcal{H}(L,1) = \Big(\coprod_{k\in\mathbb{Z}} \ell(k) \Big) \oplus \mathbb{C}c$ with multiplication

$$\text{(1.1.4)} \qquad \begin{cases} [a(k),b(\ell)] = k\delta_{k+\ell,0}(a|b)c \\ c \text{ central} \end{cases}$$

Proof: From the proof of proposition 1.1.1, we have (in $\mathcal{H}(L,n)$)

$$(1.1.5) \qquad z_k(k+\ell) = \begin{cases} 0 & , k+\ell \neq 0 \\ kz_1(0) & , k+\ell = 0 \end{cases}$$

and $\mathfrak{z} = \mathbb{C}z_1(0)$. Putting $c := z_1(0)$ we have $\mathfrak{z} = \mathbb{C}c$ and hence by (1.1.2), $\mathcal{H}(L,1) = \left(\coprod_{k\in\mathbb{Z}} \ell(k)\right) \oplus \mathbb{C}c$ as required. Now (1.1.4) follows immediately by substituting (1.1.5) into (1.1.3). ∎

Notation: In the sequel we will denote $\mathcal{H}(L,1)$ by $\mathfrak{a}(L)$.

Examples: **(1)** Suppose n=1 and L is a positive definite rank 1 geometric lattice. We consider $\mathfrak{a}(L)$. Let $0 \neq \alpha \in L$. Then $(\alpha|\alpha) \neq 0$. Put $x := \frac{\alpha}{\sqrt{(\alpha|\alpha)}} \in \ell$ so that $(x|x) = 1$. Now since ℓ is one dimensional $\ell = \mathbb{C}x$ and $\ell(k) = \mathbb{C}\cdot x(k)$, $k\in\mathbb{Z}$. Denoting $x(k)$ by x_k, we see that $\mathfrak{a}(L)$ has generators $\{x_k, c : k\in\mathbb{Z}\}$ and relations

$$\begin{cases} [x_k, x_\ell] = k\delta_{k+\ell,0}c \\ c \text{ is central.} \end{cases}$$

This is the well-known infinite Heisenberg algebra which gives the canonical commutation relations of quantum physics.

(2) Suppose n=1 and L is nondegenerate and M is a degenerate lattice. Then $\mathfrak{a}(L) \not\cong \mathfrak{a}(M)$. This follows from proposition 1.1.1 since the centre of $\mathfrak{a}(L)$ is finite dimensional but the centre of $\mathfrak{a}(M)$ is infinite dimensional. Let us focus on an explicit example. Let $L = \dot{Q}$ be a lattice of type A_ℓ, D_ℓ or E_ℓ. Then L is a positive definite and hence nondegenerate lattice. Thus the centre of $\mathfrak{a}(L)$ is $\dot{\mathfrak{h}}(0) \oplus \mathbb{C}c$ where $\dot{\mathfrak{h}} := \mathbb{C}\otimes_{\mathbb{Z}}\dot{Q}$. Next let $Q := \dot{Q} \oplus \mathbb{Z}\delta$ where $(\dot{Q}|\delta) = 0 = (\delta|\dot{Q})$ and $(\delta|\delta) = 0$. Let $\mathfrak{h} := \mathbb{C}\otimes_{\mathbb{Z}}Q$. Consider $\mathfrak{a}(Q)$. By proposition 1.1.1, the centre of $\mathfrak{a}(Q)$ is $\mathfrak{z} \oplus \mathfrak{h}(0) \oplus \left(\coprod_{n\in\mathbb{Z}\setminus\{0\}} \mathbb{C}\delta(n)\right)$. We will spend considerable effort studying $\mathfrak{a}(Q)$ in the sequel - see chapter 3.

(3) Let L and L' be two nondegenerate lattices of rank m. Then $\mathcal{H}(L,n) \cong \mathcal{H}(L',n)$ for any $n\geq 1$. Indeed, let $\{a_i\}_{i=1}^m$ and $\{a_i'\}_{i=1}^m$ be orthonormal bases for $\ell = \mathbb{C}\otimes_{\mathbb{Z}}L$ and $\ell' := \mathbb{C}\otimes_{\mathbb{Z}}L'$ respectively. By (1.1.3), $[a_i(r),a_j(s)] = \delta_{ij}\, z_r(r+s)$ and $[a_i'(r),a_j'(s)] = \delta_{ij}z_r(r+s)$ and hence the unique linear map $\phi : \mathcal{H}(L,n) \longrightarrow \mathcal{H}(L',n)$ satisfying $\phi|_{\mathfrak{z}} \equiv \mathrm{id}$. and $a_i(r) \longmapsto a_i'(r)$ is a Lie algebra isomorphism.

Remark: Since any two lattices of the same rank have isomorphic complexifications this example suggests that one could formulate the definition of $\mathcal{H}(L,n)$ by beginning with a $\mathbb{C}$-space instead of a lattice. However, it is both customary and convenient to work with lattices.

§1.2 A Realization of $\mathcal{H}(\dot{Q},n)$

In this section we give a concrete realization of $\mathcal{H}(L,n)$ where L is the root lattice of a simple finite dimensional Lie algebra $\dot{\mathfrak{g}}$ over $\mathbb{C}$. We will assume some well-known results on central extensions of Lie algebras. We summarize these results in appendix B.

Let $\dot{\mathfrak{g}}$ be a simple finite dimensional Lie algebra over $\mathbb{C}$ with corresponding root lattice $\dot{Q}$. Let A be any commutative algebra over $\mathbb{C}$. Consider the Lie algebra $\tilde{\mathfrak{g}} := \dot{\mathfrak{g}} \otimes_{\mathbb{C}} A$ with bracket $[x\otimes a, y\otimes b] = [x,y]\otimes ab$, $x,y \in \dot{\mathfrak{g}}$ and $a,b \in A$. Here we have used $[\cdot,\cdot]$ to denote the bracket in both $\dot{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}$. Note that if A has an identity element then $\tilde{\mathfrak{g}}$ is perfect since $[\tilde{\mathfrak{g}},\tilde{\mathfrak{g}}] \supset [\dot{\mathfrak{g}}\otimes_{\mathbb{C}} A\ ,\ \dot{\mathfrak{g}}\otimes_{\mathbb{C}} \mathbb{C}] = \dot{\mathfrak{g}}\otimes_{\mathbb{C}} A = \tilde{\mathfrak{g}}$ and hence it has a unique universal covering algebra (uca). If A has no identity element then $\tilde{\mathfrak{g}}$ is still perfect but a more subtle argument is necessary. For this we refer the reader to Kassel [Ka]. The structure of the uca of $\dot{\mathfrak{g}} \otimes_{\mathbb{C}} A$ has been worked out in [Ka]. We review the construction here.

Let $\{a_i\}$ be a $\mathbb{C}$-basis of A and let F be a free left A-module with basis $\{\tilde{da}_i\}$ where $\{\tilde{da}_i\}$ is some set equipotent with $\{a_i\}$. Let $\tilde{d} : A \longrightarrow F$ be the $\mathbb{C}$-linear map $\sum_i c_i a_i \longmapsto \sum_i c_i \tilde{da}_i$. Make F into a right A-module by defining $(\tilde{da})\cdot b := b\cdot(\tilde{da})$, $a,b\in A$. Then F becomes an A-bimodule since A is commutative. Indeed, $(b\cdot(\tilde{da}))\cdot c = c\cdot(b\cdot(\tilde{da})) = cb\cdot(\tilde{da}) = bc\cdot(\tilde{da}) = b\cdot(c\cdot(\tilde{da})) = b\cdot((\tilde{da})\cdot c)$. Let K be the A-submodule of F generated by the elements

$$\tilde{d}(ab) - \big((\tilde{da})b + a(\tilde{db})\big), \quad a,b\in A.$$

Let $\Omega_A := F/K$ and $d: A \longrightarrow \Omega_A$ the map given by $da := \tilde{da} + K$.

Ω_A is called the A-*module of differentials* and d the *differential map*. Finally, let $\bar{} : \Omega_A \longrightarrow \Omega_A/dA$ be the canonical map.

Then for $a,b \in A$ we have $\overline{d(ab)} = 0$. Although we will not use it, we point out the universal property that the pair (Ω_A,d) satisfies. For every A-module M and every derivation $D : A \longrightarrow M$ there is a unique A-module homomorphism $f : \Omega_A \longrightarrow M$ such that $f \circ d = D$. Thus ones has $Der_{\mathbb{C}}(A,M) \cong Hom_A(\Omega_A,M)$.

Theorem (Kassel): The Lie algebra $\mathfrak{g} := (\dot{\mathfrak{g}} \otimes_{\mathbb{C}} A) \oplus \Omega_A/dA$ with multiplication defined by

$$(1.2.1) \qquad \begin{cases} [x\otimes a, y\otimes b] := [x,y]\otimes ab + (x|y)\overline{(da)b} \\ \Omega_A/dA \quad \text{central} \end{cases}$$

is the universal covering algebra of $\dot{\mathfrak{g}}\otimes_{\mathbb{C}}A$. Here $(\cdot|\cdot)$ denotes the Killing form on $\dot{\mathfrak{g}}$.

Proof: Indeed, the short exact sequence

$$0 \longrightarrow \Omega_A/dA \xrightarrow{\ i\ } \underbrace{(\dot{\mathfrak{g}}\otimes_{\mathbb{C}}A) \oplus \Omega_A/dA}_{\mathfrak{g}} \xrightarrow{\ \pi\ } \underbrace{\dot{\mathfrak{g}}\otimes_{\mathbb{C}}A}_{\tilde{\mathfrak{g}}} \longrightarrow 0$$

where i denotes the canonical embedding and π the canonical projection is a central extension of $\tilde{\mathfrak{g}}$ since $i\left(\Omega_A/dA\right)$ is central in $\mathfrak{g}$. Moreover, since $(\cdot|\cdot)$ is nondegenerate on $\dot{\mathfrak{g}}$, $\mathfrak{g}$ is perfect and hence the central

extension is a covering of $\tilde{\mathfrak{g}}$. Thus to show that it is the universal covering algebra we need only show that there exists a morphism from it to any other central extension to $\mathfrak{g}$. This is a more subtle issue and we refer the reader to [MEY] for details. ∎

Remark: We examine Ω_A when $A = \mathbb{C}[t_1^{\pm 1},\dots,t_n^{\pm 1}]$. In this case F is freely generated as an A-module by

$$\left\{ \tilde{d}(t_1^{r_1}\cdots t_n^{r_n}) \;:\; r = (r_1,\dots,r_n) \in \mathbb{Z}^n \right\}.$$

Thus Ω_A is generated by $\left\{ d(t_1^{r_1}\cdots t_n^{r_n}) : r \in \mathbb{Z}^n \right\}$ over A but not freely since, for example, $\tilde{d}(t_1t_2)-(\tilde{dt_1})t_2-t_1(\tilde{dt_2}) \in K$ and hence we have the nontrivial relation $d(t_1t_2)-(dt_1)t_2-t_1(dt_2) = 0$ in Ω_A. However, Ω_A is freely generated over A by $\{dt_i : i=1,\dots,n\}$ and hence over $\mathbb{C}$ by $\left\{ t_1^{r_1}\cdots t_i^{r_i-1}\cdots t_n^{r_n}\, dt_i \;:\; 1\le i\le n,\ r\in\mathbb{Z}^n \right\}$. When $A = \mathbb{C}[t_1^{\pm},\dots,t_n^{\pm}]$ we denote the algebra $\mathfrak{g}$ simply by $\tau_{[n]}$ and we call it the *toroidal algebra*.

Proposition 1.2.1: Let $\dot{\mathfrak{g}}$ be a simple finite dimensional Lie algebra over $\mathbb{C}$ with root lattice $\dot{Q}$ and fixed Cartan subalgebra $\dot{\mathfrak{h}}$. Let $\mathcal{A}$ be the subalgebra of $\tau_{[n]}$ generated by the subspace $\dot{\mathfrak{h}} \otimes_{\mathbb{C}} \mathbb{C}[t_1^{\pm 1},\dots,t_n^{\pm 1}]$. Then, as Lie algebras, $\mathcal{H}(\dot{Q},n) \cong \mathcal{A}$.

Proof: First observe from (1.2.1) that, as a $\mathbb{C}$-space, $\mathcal{A} = \dot{\mathfrak{h}} \otimes \mathbb{C}[t_1^{\pm 1},\dots,t_n^{\pm 1}] \oplus \Omega_A/dA$. Let $\hat{\mathfrak{h}} := \mathbb{C}\otimes_{\mathbb{Z}}\dot{Q}$. Our first goal is to exhibit a linear isomorphism $\phi : \left(\coprod_{r\in\mathbb{Z}^n} \hat{\mathfrak{h}}(r) \right) \oplus 3 \to \left(\dot{\mathfrak{h}} \otimes_{\mathbb{C}} \mathbb{C}[t_1^{\pm 1},\dots,t_n^{\pm 1}] \right) \oplus \Omega_A/dA$. As $\mathbb{C}$-spaces, $\hat{\mathfrak{h}} \cong \dot{\mathfrak{h}}$ so we identify them and always write $\dot{\mathfrak{h}}$ with the

understanding that in $\mathcal{H}(\dot{Q},n)$ only the underlying $\mathbb{C}$-space structure of $\dot{\mathfrak{h}}$ is considered. Since $\dot{\mathfrak{h}} \otimes_{\mathbb{C}} \mathbb{C}[t_1^{\pm 1},\dots,t_n^{\pm 1}] = \coprod_{r\in\mathbb{Z}^n} \left(\dot{\mathfrak{h}} \otimes_{\mathbb{C}} \mathbb{C}t_1^{r_1}\cdots t_n^{r_n} \right)$ and $\dot{\mathfrak{h}}(r)$ is by definition an isomorphic copy of $\dot{\mathfrak{h}}$, it is clear that the unique linear map $\phi_1 : \coprod_{r\in\mathbb{Z}^n} \dot{\mathfrak{h}}(r) \to \dot{\mathfrak{h}} \otimes_{\mathbb{C}} \mathbb{C}[t_1^{\pm 1},\dots,t_n^{\pm 1}]$ extending

$$\gamma(r) \longmapsto \gamma \otimes (t_1^{r_1}\cdots t_n^{r_n}), \quad \gamma\in\dot{\mathfrak{h}},\ r=(r_1,\dots,r_n)\in\mathbb{Z}^n$$

is an isomorphism.

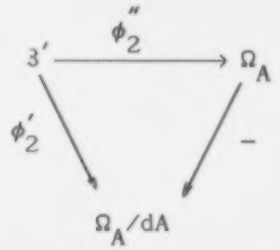
Next let $\{e_i := (0,\dots,1,\dots 0) : 1\le i\le n\}$ be the standard basis for $\mathbb{C}^n$. Then a basis for $\mathfrak{Z}' = \left(\coprod_{r\in\mathbb{Z}^n} \mathfrak{Z}'(r) \right)$ is given by $\left\{ z_{e_i}(r) : 1\le i\le n,\ r\in\mathbb{Z}^n \right\}$.

By the remark preceding this proposition, we know that

$\left\{ t_1^{r_1}\cdots t_i^{r_i-1}\cdots t_n^{r_n}\, dt_i : 1\le i\le n,\ r\in\mathbb{Z}^n \right\}$ is a $\mathbb{C}$-basis for Ω_A. Thus the unique linear map $\phi_2'' : \mathfrak{Z}' \to \Omega_A$ extending

$$z_{e_i}(r) \longmapsto t_1^{r_1}\cdots t_i^{r_i-1}\cdots t_n^{r_n}\, dt_i \tag{1.2.2}$$

is an isomorphism. Now ϕ_2'' induces a surjective linear map $\phi_2' : \mathfrak{Z}' \to \Omega_A/dA$ such that the following diagram commutes.



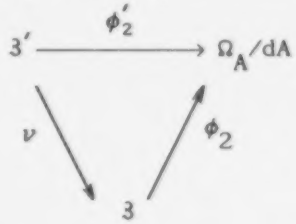
where - denotes the canonical projection.

Now $z_s(r) \in \ker \phi_2' \iff \sum_{i=1}^{n} s_i\, t_1^{r_1}\cdots t_i^{r_i-1}\cdots t_n^{r_n}\, dt_i \in dA.$

But dA is spanned by elements of the form

$d(t_1^{r_1}\cdots t_n^{r_n}) = \sum_{i=1}^{n} r_i\, t_1^{r_1}\cdots t_i^{r_i-1}\cdots t_n^{r_n}\, dt_i$, $r\in\mathbb{Z}^n$ and since

$\left\{ t_1^{r_1}\cdots t_i^{r_i-1}\cdots t_n^{r_n}\, dt_i : 1\le i\le n,\ r\in\mathbb{Z}^n \right\}$ is a linearly independent set we have $z_s(r) \in \ker \phi_2' \iff r = s$. Thus by linearity we have $\ker \phi_2' = R$. Consequently there exists a unique linear map $\phi_2 : \mathfrak{Z} \to \Omega_A/dA$ such that the diagram



commutes where $\nu : \mathfrak{Z}' \longrightarrow \mathfrak{Z}$ is the canonical projection. Thus the map $\phi : \mathcal{H}(\mathring{Q},n) \to \mathcal{A}$ defined by $\phi := \phi_1 \oplus \phi_2$ is a linear isomorphism and for $r = (r_1,\ldots,r_n)$, $s = (s_1,\ldots,s_n) \in \mathbb{Z}^n$ we have

$$[\phi(\gamma(r)),\phi(\eta(s))] = [\gamma\otimes(t_1^{r_1}\cdots t_n^{r_n}) ,\ \eta\otimes(t_1^{s_1}\cdots t_n^{s_n})]$$

$$= (\gamma|\eta)\left\{ \overline{d(t_1^{r_1}\cdots t_n^{r_n})t_1^{s_1}\cdots t_n^{s_n}} \right\}$$

$$= (\gamma|\eta)\left\{ \sum_{i=1}^{n} \overline{r_i\, t_1^{r_1+s_1}\cdots t_i^{r_i+s_i-1}\cdots t_n^{r_n+s_n}\, dt_i} \right\}$$

$$= (\gamma|\eta)\ \phi_2\big(z_r(r+s)\big), \text{ by (1.2.2)}$$

$$= \phi_2\big((\gamma|\eta)z_r(r+s)\big)$$

$$= \phi\big([\gamma(r),\eta(s)]\big).$$

Thus ϕ preserves the Lie bracket and hence is a Lie algebra isomorphism. This completes the proof. ∎

§1.3 Sub - Triangular Decomposition

In the last section we considered the generalized Heisenberg algebras $\mathcal{H}(L,n)$ where L is the root lattice $\dot{Q}$ of a simple finite dimensional Lie algebra $\dot{\mathfrak{g}}$ over $\mathbb{C}$. In this section we fix n=2 and when $\dot{Q}$ is of type A_ℓ, D_ℓ or E_ℓ we denote the algebra $\mathcal{H}(\dot{Q},2)$ simply by $\mathcal{H}$. Although $\mathcal{H}$ does not admit a triangular decomposition in the sense of [MP-2], it has a subalgebra $\mathfrak{L}$, which after suitable extension by derivations, does admit a triangular decomposition. This subalgebra $\mathfrak{L}$ of $\mathcal{H}$ is isomorphic to the Heisenberg algebra $\mathfrak{a}(Q)$ discussed in example 2 in §1. We begin with a proposition which gives a basis for $\mathfrak{Z}$. One can find the analogue of this result in the Ω_A/dA-setting in [MEY].

Proposition 1.3.1: The collection of elements

$$\mathfrak{B} := \left\{ z_{(0,1)}(m,0),\ z_{(1,0)}(0,0) : m\in\mathbb{Z} \right\} \cup$$

$$\left\{ z_{(1,0)}(m,n) : m\in\mathbb{Z},\ n\in\mathbb{Z}\setminus\{0\} \right\} \text{ is a basis for } \mathfrak{Z} = \mathfrak{Z}'/R \text{ over } \mathbb{C}.$$

Proof: We begin with the full set of elements $z_{(r,s)}(m,n) \in \mathfrak{Z}'$, r,s,m,n $\in \mathbb{Z}$ which, by (1.1.1), are linear in the subscript (r,s) for each given (m,n). Thus we may restrict our attention to the collection of elements $\mathfrak{B}' := \left\{ z_{(1,0)}(m,n),\ z_{(0,1)}(m,n) : m,n \in \mathbb{Z} \right\}$. In the quotient space $\mathfrak{Z} = \mathfrak{Z}'/R$ one has the relations

$$m\, z_{(1,0)}(m,n) + n\, z_{(0,1)}(m,n) = 0\ ,\quad m,n \in \mathbb{Z}. \tag{1.3.1}$$

For $n\neq 0$ we have $z_{(0,1)}(m,n) = -(m/n)\, z_{(1,0)}(m,n)$ and so we remove the subcollection $\left\{ z_{(0,1)}(m,n) : m\in\mathbb{Z},\ n\in\mathbb{Z}\setminus\{0\} \right\}$ from $\mathfrak{B}'$. We are left with

$\mathfrak{B}'' := \left\{ z_{(1,0)}(m,n),\ z_{(0,1)}(m,0) : m,n\in\mathbb{Z} \right\}$. Finally if n=0 then the equations in (1.3.1) read $m\left(z_{(1,0)}(m,0)\right) = 0$, $\forall m\in\mathbb{Z}$. Removing the elements $z_{(1,0)}(m,0)$, $m\neq 0$ from $\mathfrak{B}''$ we are left with a set which, of course, still generates 3 and it is the proposed basis set $\mathfrak{B}$. Since the elements of $\mathfrak{B}$ were linearly independent in $3'$ and by inspection no linear combination of them can lie in R it is clear that they are also linearly independent in 3. ∎

Introduce the following subspaces of $\mathcal{H} = \left(\coprod_{(m,n)\in\mathbb{Z}^2} \ell(m,n) \right) \oplus 3$:

$$\mathfrak{L} := \left(\coprod_{m\in\mathbb{Z}} \ell(m,0) \right) \oplus \left(\coprod_{m\in\mathbb{Z}} \mathbb{C}z_{(0,1)}(m,0) \right) \oplus \left(\mathbb{C}z_{(1,0)}(0,0) \right) \quad \text{and}$$

$$\mathfrak{M} := \coprod_{\substack{m,n\in\mathbb{Z} \\ n\neq 0}} \left(\ell(m,n) \oplus \mathbb{C}z_{(1,0)}(m,n) \right).$$

Proposition 1.3.2: As $\mathbb{C}$-spaces, $\mathcal{H} = \mathfrak{L} \oplus \mathfrak{M}$ and $\mathfrak{L}$ is a subalgebra of $\mathcal{H}$ isomorphic to $\mathfrak{a}(Q)$.

Proof: The fact that $\mathcal{H} = \mathfrak{L} \oplus \mathfrak{M}$ follows from proposition 1.3.1. Thus it suffices to check that $\mathfrak{L}$ is a subalgebra of $\mathcal{H}$ isomorphic to $\mathfrak{a}(Q)$. For $\gamma,\eta \in \ell$ and $m,n \in \mathbb{Z}$ we have $[\gamma(m,0),\eta(n,0)] = (\gamma|\eta)\ z_{(m,0)}(m+n,0)$

$$= m(\gamma|\eta)\ z_{(1,0)}(m+n,0)$$

$$= m\delta_{m+n,0}(\gamma|\eta)\ z_{(1,0)}(0,0)$$

where the last equality above follows from the fact that if $m+n\neq 0$ then $z_{(1,0)}(m+n,0) = \frac{1}{m+n} z_{(m+n,0)}(m+n,0) \in R$ and hence $z_{(1,0)}(m+n,0) = 0$ in $\mathfrak{Z} = \mathfrak{Z}'/R$. Now recall the Heisenberg algebra $\mathfrak{a}(Q)$ from example 1 in §1. It is clear that under the correspondences

$$\begin{cases} \gamma(m,0) \longleftrightarrow \gamma(m) \\ z_{(0,1)}(m,0) \longleftrightarrow \delta(m) \\ z_{(1,0)}(0,0) \longleftrightarrow c \end{cases}$$

we have $\mathfrak{L} \cong \mathfrak{a}(Q)$. ∎

Note that $\mathfrak{M}$ is not a subalgebra since, for example, if $m\in\mathbb{Z}$ and $\gamma,\eta \in \ell$ then $[\gamma(0,1),\eta(m,-1)] = (\gamma|\eta)\, z_{(0,1)}(m,0) \in \mathfrak{L}$.

§1.4 The Canonical Representation

Motivated by our discussions about the harmonic oscillator in the main introduction of the thesis, we introduce here the canonical representation of $\mathfrak{a}(L)$ where L is any geometric lattice. We will frequently vary the lattice and use both nondegenerate and degenerate ones.

In §1 we met the Heisenberg algebra $\mathfrak{a}(L) = \left(\coprod_{n\in\mathbb{Z}} \ell(n) \right) \oplus \mathbb{C}c$ with commutation relations given by (1.1.4). Observe that $\ell(0)$ is an abelian subalgebra of $\mathfrak{a}(L)$ and its complement $\mathring{\mathfrak{a}}(L) := \left(\coprod_{n\in\mathbb{Z}\setminus\{0\}} \ell(n) \right) \oplus \mathbb{C}c$ is a subalgebra of $\mathfrak{a}(L)$ satisfying $\mathfrak{a}(L) = \mathring{\mathfrak{a}}(L) \times \ell(0)$ where $\times$ denotes the direct product of Lie algebras. The canonical representation of $\mathfrak{a}(L)$ is constructed by first defining an auxiliary representation of the subalgebra $\mathring{\mathfrak{a}}(L)$.

Let $\mathfrak{a}(L)_- := \coprod_{n<0} \ell(n)$ and let $S(\mathfrak{a}(L)_-)$ be the corresponding symmetric algebra. We call $S(\mathfrak{a}(L)_-)$ the *Fock space* corresponding to L.

Fix a basis $\{a_i\}_{i=1}^m$ of $\ell = \mathbb{C}\otimes_{\mathbb{Z}} L$. Then we may think of $S(\mathfrak{a}(L)_-)$ as the polynomial ring in the indeterminates $\{a_i(-n) : a_i\in L,\ 1\le i\le m,\ n>0\}$ and any monomial in $S(\mathfrak{a}(L)_-)$ is an element of the form $x = a_{i_1}(-n_1)\cdots a_{i_k}(-n_k)$ where $1\le i_j\le m$ and $n_j>0$ for $j=1,\dots,k$. We say that x has *length* k and $S(\mathfrak{a}(L)_-)$-*degree* $n_1+\cdots+n_k$.

For $r,s>0$ let $\mathbf{n}=(n_1,\dots,n_s)\in\mathbb{Z}_+^s$ and $\mathbf{m}=(m_1,\dots,m_r)\in\mathbb{Z}_+^r$, $\mathbf{m}\neq\mathbf{n}$. Suppose $s\ge r$. View $\mathbf{m}\in\mathbb{Z}_+^s$ by writing $\mathbf{m}=(m_1,\dots,m_r,\underbrace{0,\dots,0}_{s-r \text{ times}})$. We say that $\mathbf{n}>\mathbf{m}$ if $n_i>m_i$ for the first i such that $n_i\neq m_i$. Thus the set of elements $\mathbf{n} \in \bigcup_{s>0} \mathbb{Z}_+^s$ is totally ordered. This is called the lexicographical ordering.

For $\mathbf{a} := (a_{i_1},\dots,a_{i_k})$, $a_{i_j}\in \ell$ and $\mathbf{n}_i=(n_{i_1},\dots,n_{i_k}) \in \mathbb{Z}_+^k$ we denote the element

$a_{i_1}(n_{i_1})\cdots a_{i_k}(n_{i_k}) \in \mathfrak{U}(\mathfrak{a}(L))$ by $\mathbf{a_i}(\mathbf{n_i})$.

Let $\mathring{\phi} : \mathring{\mathfrak{a}}(L) \longrightarrow \mathfrak{gl}\left(S(\mathfrak{a}(L)_-)\right)$ be the unique linear map satisfying

$$(1.4.1) \qquad \begin{cases} c \longmapsto I \\ a(-n) \longmapsto \ell_{a(-n)} \\ a(n) \longmapsto \partial_{a(n)}, \quad \text{for } a\in\ell,\ n>0 \end{cases}$$

where $\ell_{a(-n)}f := a(-n)f$ for any $f\in S(\mathfrak{a}(L)_-)$; $\partial_{a(n)}$ is the unique derivation of $S(\mathfrak{a}(L)_-)$ satisfying $\partial_{a(n)}(b(-m)) := n\delta_{n,m}(a|b)$, $b\in\ell$, $m>0$ and $\partial_{a(n)}(c) := 0$.

Proposition 1.4.1: $\mathring{\phi} : \mathring{\mathfrak{a}}(L) \longrightarrow \mathfrak{gl}\left(S(\mathfrak{a}(L)_-)\right)$ as defined above gives a representation of $\mathring{\mathfrak{a}}(L)$ on $S(\mathfrak{a}(L)_-)$.

Proof: It suffices to show

$$(1.4.2) \qquad \mathring{\phi}[a(n),b(m)] = [\mathring{\phi}(a(n)),\mathring{\phi}(b(m))]$$

We need only examine the following four mutually exclusive cases:

(i) $n<0,\ m<0$

(ii) $n<0,\ m>0,\ n+m\neq 0$

(iii) $n>0,\ m>0$

(iv) $n<0,\ m>0,\ n+m=0$

In each of the first three cases, the left side of (1.4.2) is zero by (1.1.4) since $n+m\neq 0$. In case (i), the right side of (1.4.2) is zero since multiplication is commutative in $S(\mathfrak{a}(L)_-)$. In (ii), the right side is zero

since $\ell_{a(n)}$ commutes with $\partial_{b(m)}$ if $n+m\neq 0$. In case (iii), one shows that the right side is zero when applied to any monomial $x \in S(\mathfrak{a}(L)_-)$ by induction on the length of x. This requires only the product rule property of a derivation.

Finally, in case (iv), the left side is

$\mathring{\phi}[a(n),b(m)] = \mathring{\phi}(n(a|b)c) = n(a|b)$.

As for the right side, for any $x \in S(\mathfrak{a}(L)_-)$ we have

$$\begin{aligned}[\mathring{\phi}(a(n)),\mathring{\phi}(b(m))](x) &= \ell_{a(n)}\partial_{b(m)}(x) - \partial_{b(m)}\big(\ell_{a(n)}(x)\big) \\ &= \ell_{a(n)}\partial_{b(m)}(x) - \Big\{\partial_{b(m)}\big(a(n)\big)x + \ell_{a(n)}\partial_{b(m)}(x)\Big\} \\ &= -\partial_{b(m)}\big(a(n)\big)x \\ &= -m(a|b)\delta_{n+m,0}x \\ &= n(a|b)x\end{aligned}$$

This shows that $\mathring{\phi}$ is a representation. ∎

We call $S(\mathfrak{a}(L)_-)$ the *canonical representation* of $\mathring{\mathfrak{a}}(L)$.

Proposition 1.4.2: For the representation $\mathring{\phi}$, the following are equivalent:

(i) $\mathring{\phi}$ is faithful

(ii) L is nondegenerate

(iii) $S(\mathfrak{a}(L)_-)$ is irreducible over $\mathring{\mathfrak{a}}(L)$.

Proof: (i) $\Leftrightarrow$ (ii): $\mathring{\phi}$ is injective

$\Leftrightarrow$ $\{a(n),\ n>0 : \partial_{a(n)} \equiv 0\} = \{0\}$

$\Leftrightarrow$ $\{a\in L : (a|b) = 0\ \forall b\in L\} = \{0\}$

$\Leftrightarrow$ L is nondegenerate.

(iii) ⇒ (ii) Suppose L is degenerate and R is the radical of the form $(\cdot|\cdot)$ on L. Let $r := \mathbb{C}\otimes_{\mathbb{Z}}R$ and let r(n) be an isomorphic copy of r for every $n\in\mathbb{Z}$. Define $a(R)_- := \coprod_{n<0} r(n)$ and let $S(a(R)_-)$ be its symmetric algebra. Then for any ideal E of $S(a(R)_-)$, it is clear that, since the elements in $a(R)_-$ are central in a(L), the subspace $E\cdot S(a(L)_-)$ of $S(a(L)_-)$ is an $\mathring{a}(L)$-submodule.

(ii) ⇒ (iii) Let $\{a_i\}_{i=1}^m$ be an orthonormal basis of $\ell = \mathbb{C}\otimes_{\mathbb{Z}}L$ over $\mathbb{C}$. Since $S(a(L)_-)$ is the polynomial ring in the indeterminates $\{a_i(-n) : 1\le i\le m,\ n>0\}$ and $\partial_{a_i(n)}(a_j(-m)) = n(a_i|a_j)\delta_{i,j}\delta_{n,m}$ we see that the derivation $\partial_{a_i(n)}$ corresponds to the partial differentiation operator $n\dfrac{\partial}{\partial(a_i(-n))}$ on $S(a(L)_-)$.

Since $S(a(L)_-)$ is a cyclic $\mathring{a}(L)$-module generated by 1 it suffices to show that any nontrivial submodule N of $S(a(L)_-)$ contains 1. Let $0\ne f\in N$. Write

$$f = \sum_{i=1}^{t} a_i(-n_i) \text{ with } n_i := (n_{i_1},\dots,n_{i_{p_i}}) \in \mathbb{Z}_+^{p_i},\ a_i := (a_{i_1},\dots,a_{i_{p_i}}),$$

$a_{i_j}\in\ell$ for some $p_1>0$ and $n_1>n_2>\cdots>n_t$ with respect to the lexicographical ordering. Then $f = a_1(-n_1) + f'$ where $f' = \sum_{i=2}^{t} a_i(-n_i)$. Now since L is nondegenerate there exists $b_i\in\ell$ such that $(a_i|b_i)\ne 0$ for all i. Denoting $\partial_{b_{i_1}(n_{i_1})}\cdots\partial_{b_{i_{p_1}}(n_{i_{p_1}})}$ by $\partial_{b_i(n_i)}$ we have $\partial_{b_1(n_1)}\cdot f' = 0$ while $\partial_{b_1(n_1)}a_1(-n_1)\in\mathbb{C}^\times$. ∎

We now turn to the problem of extending the representation of $\mathring{a}(L)$ on $S(a(L)_-)$ to a representation of $a(L) = \mathring{a}(L)\times\ell(0)$ on some isomorphic copy of $S(a(L)_-)$. Since (by prop 1.1.1) all elements in $\ell(0)$ are central in a(L), the

natural way to represent them is by scalars operators. We do this in the following way.

Let M be any nondegenerate lattice containing L. One may choose M=L if L is already nondegenerate. Put $m := \mathbb{C}\otimes_{\mathbb{Z}}M$ and fix $\lambda\in m$. Let $\mathbb{C}e^{\lambda}$ be the one-dimensional space spanned by the symbol e^{λ}.

Consider the $\mathbb{C}$-space:

$$V_L(\lambda) := \mathbb{C}e^{\lambda}\otimes_{\mathbb{C}}S(\mathfrak{a}(L)_-). \tag{1.4.3}$$

Of course, as $\mathbb{C}$-spaces we have $V_L(\lambda) \cong S(\mathfrak{a}(L)_-)$. We make $V_L(\lambda)$ into an $\mathfrak{a}(L)$-module by defining

$$\begin{cases} a(n)\cdot(e^{\lambda}\otimes f) := e^{\lambda}\otimes a(n)\cdot f, \ n\neq 0 \\ a(0)\cdot(e^{\lambda}\otimes f) := (a|\lambda)\, e^{\lambda}\otimes f \\ c\cdot(e^{\lambda}\otimes f) := (e^{\lambda}\otimes f) \end{cases} \tag{1.4.4}$$

where $a(n)\cdot f$, $n\neq 0$, is given by the $\mathring{\mathfrak{a}}(L)$-module $S(\mathfrak{a}(L)_-)$ in (1.4.1).

In the sequel we will denote the operator $\ell_{a(-n)}$ by $a(-n)$ and $\partial_{a(n)}$ by $a(n)$. The representation $V_L(\lambda)$ is called the *canonical representation* of $\mathfrak{a}(L)$.

Proposition 1.4.3: $V_L(\lambda)$ is an irreducible $\mathfrak{a}(L)$-module if and only if L is a nondegenerate lattice.

Proof: The proof is the same as that in proposition 1.4.2 (iii) $\Leftrightarrow$ (ii) with the obvious notational changes. For example, one shows that any nonzero polynomial $f\in V_L(\lambda)$ can be reduced to $e^{\lambda}\otimes 1$ instead of 1 as in the case of $S(\mathfrak{a}(L)_-)$. ∎

Remarks: (i) Note that contrary to the $\mathring{\mathfrak{a}}(L)$-module $S(\mathfrak{a}(L)_-)$, $V_L(\lambda)$ is never a faithful $\mathfrak{a}(L)$-module.

(ii) We will frequently apply proposition 1.4.3 in the case that L is of type A_ℓ, D_ℓ or E_ℓ.

Example : (Continued from §1) Let $L = Q = \dot{Q} \oplus \mathbb{Z}\delta$ and let $\Gamma := \dot{Q} \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\mu$ where $(\mu|\dot{Q})=0=(\dot{Q}|\mu)$, $(\mu|\mu)=0$ and $(\mu|\delta)=1=(\delta|\mu)$. Then Γ is a nondegenerate lattice containing Q. Let $\mathfrak{h} := \mathbb{C}\otimes_{\mathbb{Z}}\Gamma$, $\lambda\in\Gamma$ and consider the $\mathfrak{a}(Q)$-module $V_Q(\lambda)$. By proposition 1.4.3, $V_Q(\lambda)$ is not irreducible.

Remark: In chapter 3 we will extend $\mathfrak{a}(Q)$ in such a way that, not only does the enlarged algebra then admit a triangular decomposition in the full sense of [MP-2], but it will admit irreducible representations. If we were simply interested in obtaining a triangular decompositon of $\mathfrak{a}(Q)$ then we could proceed as in the case of Kac-Moody theory and extend $\mathfrak{a}(Q)$ by adjoining a single derivation. However the infinite dimensional centre in $\mathfrak{a}(Q)$ requires a much larger extension from the point of view of producing irreducible representations. For this reason we have chosen to extend $\mathfrak{a}(Q)$ by the full Virasoro algebra. In chapter 2, we develop a representation theory of the Virasoro algebra suitably designed to handle the representations of $\mathfrak{a}(Q)$ and later $\mathcal{H}$.

§1.5 Induced Modules and Unitarity

In this section we will identify the canonical representation of $\mathfrak{a}(L)$ by means of an induced module relative to a triangular decomposition of the algebra. Using this, we address the question of whether of not the canonical representation is unitary. We end the chapter with an example that will be important in chapter 2.

Let L be a nondegenerate geometric lattice, ℓ its complexification, $\mathfrak{a}(L)$ the corresponding Heisenberg algebra and for $\lambda\in\ell$, $V_L(\lambda)$ the canonical representation of $\mathfrak{a}(L)$. Define $\mathfrak{a}(L)_{\pm} := \coprod_{n\gtrless 0} \ell(n)$ and $\mathfrak{a}(L)_0 := \ell(0) \oplus \mathbb{C}c$. Then we have the decomposition $\mathfrak{a}(L) = \mathfrak{a}(L)_- \oplus \mathfrak{a}(L)_0 \oplus \mathfrak{a}(L)_+$ into abelian subalgebras. Next let $\sigma : \mathfrak{a}(L) \longrightarrow \mathfrak{a}(L)$ be the unique anti-linear map satisfying $\sigma(a(n)) := a(-n)$, $a\in\ell$, $n\in\mathbb{Z}$ and $\sigma(c) := c$. Then σ fixes $\mathfrak{a}(L)_0$ and $\sigma(\mathfrak{a}(L)_{\pm}) = \mathfrak{a}(L)_{\mp}$ so that σ is an anti-involution.

Let $\alpha \in \mathfrak{a}(L)_0^*$, the dual of $\mathfrak{a}(L)_0$, and consider the one-dimensional space $\mathbb{C}_\alpha := \mathbb{C}v_+$ where v_+ is some symbol. Let $\mathfrak{b} := \mathfrak{a}(L)_0 \oplus \mathfrak{a}(L)_+$ and make $\mathbb{C}_\alpha$ into a $\mathfrak{b}$-module by defining

$$(1.5.1) \qquad \begin{cases} \alpha(L)_+ \cdot v_+ \equiv 0 \\ a(0) \cdot v_+ = \alpha(a(0))v_+ \\ c \cdot v_+ = v_+. \end{cases}$$

Finally, define the induced $\mathfrak{a}(L)$-module $M(\alpha) := \mathfrak{U}(\mathfrak{a}(L))\otimes_{\mathfrak{U}(\mathfrak{b})}\mathbb{C}_\alpha$.

Proposition 1.5.1: For $\lambda\in\ell$, let $\alpha \in \mathfrak{a}(L)_0^*$ be defined by $\alpha(a(0)) := (\lambda|a)$ and $\alpha(c) := 1$. Then $V_L(\lambda) \cong M(\alpha)$ as $\mathfrak{a}(L)$-modules.

Proof: Define $\phi : V_L(\lambda) \longrightarrow M(\alpha)$ to be the unique $\mathbb{C}$-linear map satisfying $\phi(e^{\lambda}\otimes u) := u\otimes v_+$ where $u \in \mathfrak{U}(\mathfrak{a}(L)_-)$. We first show that ϕ is an $\mathfrak{a}(L)$-module map. Indeed, it suffices to consider the case $u = a_{i_1}(-k_1)\cdots a_{i_r}(-k_r)$ where $a_{i_j}\in L$ and $k_j>0$, $1\le j\le r$. Let $u_j := a_{i_1}(-k_1)\cdots\overline{a_{i_j}(-k_j)}\cdots a_{i_r}(-k_r)$ where the overbar denotes omission. Now let $\ell\in\mathbb{Z}$ and $b\in L$. We will show

$$\phi\Big(b(\ell)\cdot(e^{\lambda}\otimes u)\Big) = b(\ell)\cdot\phi(e^{\lambda}\otimes u). \tag{1.5.2}$$

When $\ell=0$ or $\ell<0$, (1.5.2) is clear. When $\ell>0$ we have

$$LS = \phi\Big(b(\ell)\cdot(e^{\lambda}\otimes u)\Big) = \phi\Big\{\sum_{j=1}^{r}(b|a_{i_j})\delta_{k_j,\ell}(e^{\lambda}\otimes u_j)\Big\} = \sum_{j=1}^{r}(b|a_{i_j})\delta_{k_j,\ell}(u_j\otimes v_+).$$

On the other hand, $RS = b(\ell)\cdot(u\otimes v_+) = b(\ell)\cdot a_{i_1}(-k_1)\cdots a_{i_r}(-k_r)\otimes v_+$

$$= [b(\ell),a_{i_1}(-k_1)\cdots a_{i_r}(-k_r)]\otimes v_+$$

$$= LS.$$

Next, since L is nondegenerate, $V_L(\lambda)$ is irreducible (prop. 1.4.3) and hence ϕ is $\ker\phi = \langle 0\rangle$ and so ϕ is injective. Finally, since ϕ takes the generator $e^{\lambda}\otimes 1$ to the generator $1\otimes v_+$, it is surjective. ∎

Corollary 1.5.2: Let $\alpha \in \mathfrak{a}(L)_0^*$ satisfy (1.5.1). Then $M(\alpha)$ is an irreducible $\mathfrak{a}(L)$-module.

Proof: This follows immediately form propositions 1.4.3 and 1.5.1.

We end this section with a discussion of the existence and uniqueness of a Hermitian contravariant form $(\cdot|\cdot)$ on $M(\alpha)$. We begin with a well-known result whose proof can be found in prop. 1.8.2 of [FLM]. This form is often called the *Shapavalov form*.

Theorem: Let $\alpha \in \mathfrak{a}(L)_0^*$ satisfy $\alpha(\sigma h) = \overline{\alpha(h)}$ where the overbar denotes complex conjugation. Then there exists a unique Hermitian form on $M(\alpha)$ (linear in the first variable) satisfying

(i) $(x\cdot v|w) = (v|\sigma(x)\cdot w)$, $x \in \mathfrak{a}(L)$ - contravariance

(ii) $(v_+|v_+) = 1$ - normalization

Definition: A module V for a Lie algebra $\mathfrak{g}$ is said to be *unitary* if it admits a Hermitian contravariant positive definite form.

Example (Hyperbolic Heisenberg)

Let $\Lambda := \mathbb{Z}\delta \oplus \mathbb{Z}\mu$ be a rank 2 geometric lattice where $(\mu|\mu)=0=(\delta|\delta)$ and $(\mu|\delta) = 1 = (\delta|\mu)$. Note that Λ is an even nondegenerate lattice. By *even* we mean $(x|x) \in 2\mathbb{Z}$, $\forall x\in\Lambda$. Put $\mathfrak{p} := \mathbb{C}\otimes_{\mathbb{Z}}\Lambda$.

Now consider $\mathfrak{a}(\Lambda)$ with basis $\{\delta(n), \mu(n), c : n\in\mathbb{Z}\}$ and commutation relations given by (1.1.4). Then $\mathfrak{a}(\Lambda)_0 = \mathbb{C}\delta(0) \oplus \mathbb{C}\mu(0) \oplus \mathbb{C}c$. Let $\alpha \in \mathfrak{a}(\Lambda)_0^*$ satisfy $\alpha(\delta(0))=0$, $\alpha(\mu(0))=1$ and $\alpha(c)=1$. By the above theorem the induced $\mathfrak{a}(\Lambda)$-module $M(\alpha)$ corresponding to the decomposition $\mathfrak{a}(\Lambda) = \mathfrak{a}(\Lambda)_-\oplus \mathfrak{a}(\Lambda)_0\oplus \mathfrak{a}(\Lambda)_+$ has a unique Hermitian contravariant form $(\cdot|\cdot)$. Note that for any $m>0$ $(\delta(-m)\cdot v_+|\delta(-m)\cdot v_+) = 0$ in which case we conclude $M(\alpha)$ is not a unitary $\mathfrak{a}(\Lambda)$-module. By proposition 1.5.1, the same is true for $V_\Lambda(\lambda)$, $\lambda\in\mathbb{Z}\delta$.

THE VIRASORO ALGEBRA AND ITS OSCILLATOR REPRESENTATIONS OVER A HYPERBOLIC LATTICE

Introduction: In chapter 1 we introduced the generalized Heisenberg algebras $\mathcal{H}(L,n)$, where L is a geometric lattice and $n\in\mathbb{Z}_+$. When n=1 we denoted $\mathcal{H}(L,n)$ by $\mathfrak{a}(L)$ and we discussed the canonical representation of $\mathfrak{a}(L)$ on the Fock space $V_L(\lambda)$, $\lambda \in \ell = \mathbb{C}\otimes_{\mathbb{Z}}L$. In this the second chapter, we will introduce the Virasoro algebra, denoted $\mathcal{V}ir$, and explore a certain class of representations of it which arise from the canonical representation of the Heisenberg algebra $\mathfrak{a}(\Lambda)$ of §1.5 where Λ is a hyperbolic lattice.

To motivate our work here, let us recall the Heisenberg algebra $\mathfrak{a}$ of Ex.1, § 1.1 with generators x_k, $k\in\mathbb{Z}$; c and relations $[x_k,x_\ell] = k\delta_{k+\ell,0}c$, $k,\ell\in\mathbb{Z}$ and c is central. Its canonical representation is (as a $\mathbb{C}$-space) a symmetric algebra often denoted B for the *bosonic space*. The generators x_k, $k\neq 0$ act on B as left multiplication when $k<0$ and a derivation when $k>0$. The central elements x_0 and c act as the scalars μI, $\mu\in\mathbb{R}$ and I respectively. As mentioned in part B of the main introduction, the algebra $\mathfrak{a}$ and its canonical representation occur naturally in physics. Moreover, we pointed out that the infinite quadratic expressions $L_k = \frac{1}{2}\sum_{j\in\mathbb{Z}} :x_{-j}x_{j+k}:$, where the x_n here are elements of the algebra $\mathfrak{a}$, occur as the Fourier components of the energy-momentum tensor in physics and they afford a representation of the Virasoro algebra on the space B. It is called the *oscillator representation* of $\mathcal{V}ir$ since it arises from the canonical representation of the Heisenberg (oscillator) algebra $\mathfrak{a}$. By virtue of its strong connection with physics, this representation of $\mathcal{V}ir$ has received the attention of many researchers. Its

structure is both subtle and intriguing and the results sketched below are due to Kac, Segal, Wakimoto-Yamada and others.

It turns out that, as a module over $\mathcal{V}ir$, B is completely reducible for all $\mu\in\mathbb{R}$ and irreducible when $\mu\in\mathbb{R}\setminus\mathbb{Z}$. This method used to prove this is to exploit the unique Hermitian contravariant positive definite form $(\cdot|\cdot)$ on B - the Shapavalov form. One shows (using this form) that the lattice of submodules is complemented. Since the irreducibles in B are highest weight modules (B is unitary) one then knows that $B \cong \coprod_i m_i L(1,h_i)$ where $L(c,h)$ denotes the unique irreducible highest weight module for $\mathcal{V}ir$ with highest weight $c,h\in\mathbb{C}$ (see §2.1) and $m_i\geq 0$ denotes the multiplicity of $L(1,h_i)$ in this decomposition.

An explicit decomposition of B into submodules and a characterization of the vacuum vectors in B has been obtained using a *boson-fermion correspondence* for $\mathcal{V}ir$ which amounts to constructing another representation of $\mathcal{V}ir$ on an anti-symmetric space F which (i) is equivalent to the representation of $\mathcal{V}ir$ on B and (ii) is a much easier place to locate vacuum vectors. Remarkably, when the vacuum vectors are translated back to the space B, they turn out to be Schur polynomials! Precise statements of these results together with some details and references are found in [KR].

Now recall the Heisenberg algebra $\mathfrak{a}(\Lambda)$ of §1.5 where $\Lambda = \mathbb{Z}\delta \oplus \mathbb{Z}\mu$ and $(\delta|\delta) = 0 = (\mu|\mu)$ and $(\delta|\mu) = 1 = (\mu|\delta)$. We saw that the canonical representation $W(\lambda) \equiv V_\Lambda(\lambda)$ is irreducible and that the Shapavalov form $(\cdot|\cdot)$ on $W_\Lambda(\lambda)$ is not positive definite. Our goal is to construct oscillator representations of $\mathcal{V}ir$ on $W(\lambda)$ and to investigate its structure. It turns out that the oscillator representations of $\mathcal{V}ir$ that we will study are a tensor

product of two oscillator representations of the type discussed above (see §2.3). However, it is important to keep in mind that our ultimate gaol is to develop a representation theory of the Virasoro-toroidal algebras $\tilde{\mathfrak{f}}_{[n]} = \mathcal{V}ir \propto \mathfrak{f}_{[n]}$ by constructing suitable representations of $\mathcal{V}ir$ that will "lift" to the semi-direct product. To this end, the tensor product decomposition just mentioned will not be of any use. Moreover, since the canonical representation of $\alpha(\Lambda)$ is not unitary, the method outlined above that is used in the well-understood case will not work here.

Beginning in section 4 we will show that our $\mathcal{V}ir$-module $W(0)$ decomposes into an infinite direct sum of submodules $\overline{W_k(0)}$, $k\in\mathbb{Z}$ and that for any $\lambda\in\mathbb{Z}\delta$, $W(\lambda)$ admits a filtration of submodules $\{W^k(\lambda),\ k\in\mathbb{Z}\}$. It turns out that for every $\lambda\in\mathbb{Z}\delta$ and every $k\in\mathbb{Z}$, we have the short exact sequence of $\mathcal{V}ir$-modules: $0 \longrightarrow W^{k-1}(\lambda) \longrightarrow W^k(\lambda) \longrightarrow \overline{W_k(0)} \longrightarrow 0$. In section 5 we introduce a positive definite Hermitian form on the submodules $\overline{W_k(0)}$, $k\in\mathbb{Z}$ and in section 6 we show that with respect to this form and the usual anti-linear anti-involution σ on $\mathcal{V}ir$ the action is contravariant. From this we use the standard argument to conclude that each $\overline{W_k(0)}$ is completely reducible and hence $W(0)$ is completely reducible. By the exact sequence above, we also conclude that for every $\lambda\in\mathbb{Z}\delta$ and every $k\in\mathbb{Z}$ the subquotient $W_k(\lambda) = W^k(\lambda)/W^{k-1}(\lambda)$ is completely reducible.

In the final section we point out a character formula for our modules from which we develop an algorithm for obtaining an explicit decomposition into irreducible Verma modules.

§2.1 The Virasoro Algebra

The Virasoro algebra, denoted $\mathcal{V}ir$, is the Lie algebra with generators d_k, $k\in\mathbb{Z}$ and $\mathfrak{z}$ and relations

$$(2.1.1)\qquad \begin{cases} [d_k, d_\ell] := (k-\ell)d_{k+\ell} + \dfrac{1}{12}\delta_{k+\ell,0}(k^3-k)\mathfrak{z}, \quad k,\ell\in\mathbb{Z} \\ \mathfrak{z} \text{ central.} \end{cases}$$

Next let $\mathcal{V}ir_+ := \coprod_{n>0} \mathbb{C}d_n$, $\mathcal{V}ir_- := \coprod_{n<0} \mathbb{C}d_n$ and $\mathcal{V}ir_0 := \mathbb{C}d_0 \oplus \mathbb{C}\mathfrak{z}$. Then $\mathcal{V}ir = \mathcal{V}ir_- \oplus \mathcal{V}ir_0 \oplus \mathcal{V}ir_+$ is a triangular decomposition of $\mathcal{V}ir$ (see [MP-2]). Let $\lambda \in \mathcal{V}ir_0^*$ and set $h:=\lambda(d_0)$ and $c:=\lambda(\mathfrak{z})$ Now form the *Verma module* M(c,h) associated to the pair (c,h). That is, if $\mathfrak{b} := \mathcal{V}ir_0 \oplus \mathcal{V}ir_+$ and $\mathbb{C}_\lambda := \mathbb{C}v_+$ where v_+ is some symbol, then we make $\mathbb{C}_\lambda$ into a $\mathfrak{b}$-module by defining

$$(2.1.2)\qquad \begin{cases} d_k \cdot v_+ := 0, \ k>0 \\ d_0 \cdot v_+ := hv_+ \\ \mathfrak{z} \cdot v_+ := cv_+ \end{cases}$$

and we form the induced $\mathcal{V}ir$-module $M(c,h) \equiv M(\lambda) := \mathcal{U}(\mathcal{V}ir)\otimes_{\mathcal{U}(\mathfrak{b})}\mathbb{C}_\lambda$. Finally let L(c,h) be the unique irreducible quotient of M(c,h). The following result is a consequence of the Kac determinant formula.

Proposition 2.1.1: M(c,h) = L(c,h) if c>1 and h>0.

Proof: See [KR] ch.8.

§2.2 Oscillator Representations of $\mathcal{V}ir$ over an arbtirary nondegenerate lattice L

Let $(L,(\cdot|\cdot))$ be an arbitrary nondegenerate geometric lattice of rank m with complexification $\ell = \mathbb{C}\otimes_{\mathbb{Z}}L$. Let $\{a_i\}_{i=1}^m$ be an orthonormal basis for ℓ over $\mathbb{C}$. Let $\mathfrak{a}(L)$ be the Heisenberg algebra and $V_L(\lambda)$, $\lambda\in L$ the Fock space associated with L on which $\mathfrak{a}(L)$ acts according to (1.4.4). Then $\mathfrak{a}(L)$ has linear basis $\{a_i(n);\ c : a_i\in L,\ 1\le i\le m,\ n\in\mathbb{Z}\}$.

For $r,s\in\mathbb{Z}$ we define a *normal ordering* of $a_i(r)a_i(s)$ by

$$:a_i(r)a_i(s): := \begin{cases} a_i(r)a_i(s), & r\le s \\ a_i(s)a_i(r), & r>s. \end{cases}$$

Now for $k\in\mathbb{Z}$ we consider the infinite quadratic expression

$$L_k := \frac{1}{2}\sum_{j\in\mathbb{Z}}\sum_{i=1}^{m} :a_i(-j)a_i(j+k): \quad . \tag{2.2.1}$$

Although for each $j\in\mathbb{Z}$ the sum $L_k(j) := \frac{1}{2}\sum_{i=1}^{m} :a_i(-j)a_i(j+k):$

belongs to $\mathfrak{U}(\mathfrak{a}(L))$, $L_k = \sum_{j\in\mathbb{Z}} L_k(j)$ does not. Due to the normal ordering, given any vector $v \in V_L(\lambda) = \mathbb{C}e^{\lambda}\otimes_{\mathbb{C}}S(\mathfrak{a}(L)_-)$, $\lambda\in\ell$ we have $L_k(j)(v) = 0$ for all but finitely many $j\in\mathbb{Z}$ and hence L_k makes sense as an operator on $V_L(\lambda)$. One should note that L_k is independent of the choice of the orthonormal basis for ℓ. Indeed, if $\{b_i\}_{i=1}^m$ is another orthonormal basis, we may write

$$a_k = \sum_{i=1}^{m} (a_k|b_i)b_i \quad \text{and hence for } r,s\in\mathbb{Z}$$

$$\sum_{k=1}^{m} a_k(r)a_k(s) = \sum_{k=1}^{m}\sum_{j=1}^{m}\sum_{i=1}^{m} (a_k|b_i)(a_k|b_j)b_i(r)b_j(s)$$

$$= \sum_{j=1}^{m}\sum_{i=1}^{m} (b_i|b_j)b_i(r)b_j(s)$$

$$= \sum_{j=1}^{m}\sum_{i=1}^{m} \delta_{i,j}b_i(r)b_j(s)$$

$$= \sum_{j=1}^{m} b_j(r)b_j(s).$$

The fundamental property of the L_k is that they afford a representation of $\mathcal{V}ir$ on $V_L(\lambda)$ on which the centre acts as mI. Showing this requires calculating the commutator $[L_k,L_\ell]$, $k,\ell\in\mathbb{Z}$. Since, by definition, L_k is given by an infinite sum, we will follow the approach used in [KR] and use the "cutoff" procedure to replace calculations with infinite sums by ones with finite sums. Define $\Psi : \mathbb{R} \longrightarrow \{0,1\rangle$ by

$$\Psi(x) := \begin{cases} 1, & |x|\leq 1 \\ 0, & |x|>1 \end{cases}$$

and put $L_k(\varepsilon) := \frac{1}{2}\sum_{j\in\mathbb{Z}}\sum_{i=1}^{m} :a_i(-j)a_i(j+k): \Psi(\varepsilon j)$. Note that if $\varepsilon\neq 0$ then $L_k(\varepsilon)$

contains only finitely many terms and for $v \in V_L(\lambda)$, we have $L_k(\varepsilon)(v) = L_k(v)$ for ε sufficiently small. Thus we write $\lim_{\varepsilon \to 0} L_k(\varepsilon) = L_k$.

Proposition 2.2.1: The assignment $\begin{cases} d_k \longmapsto L_k \\ \mathfrak{z} \longmapsto mI \end{cases}$ defines a representation of $\mathcal{V}ir$ on $V_L(\lambda)$ called the *oscillator representation* of $\mathcal{V}ir$.

Proof: We make the following important observation. For any $1 \le i \le m$

$$(2.2.2) \qquad \sum_{j \in \mathbb{Z}} a_i(-j)a_i(j+k) = \sum_{j \in \mathbb{Z}} :a_i(-j)a_i(j+k): + \sum_{j<-k/2} [a_i(-j),a_i(j+k)] ,$$

where the range of summation in the last term is over those j which are not in normal order. That is, $\{j \in \mathbb{Z} : -j > j+k\}$. Therefore, for any $1 \le r \le m$ and $n \in \mathbb{Z}$,

$$[a_r(n),L_k(\varepsilon)] = \frac{1}{2} \sum_{j \in \mathbb{Z}} \sum_{i=1}^{m} [a_r(n), :a_i(-j)a_i(j+k):] \Psi(\varepsilon j)$$

$$= \frac{1}{2} \sum_{j \in \mathbb{Z}} \sum_{i=1}^{m} [a_r(n), a_i(-j)a_i(j+k)] \Psi(\varepsilon j), \quad \text{by (2.2.2)}$$

$$= \frac{1}{2} \sum_{j \in \mathbb{Z}} \sum_{i=1}^{m} \Big\{ [a_r(n),a_i(-j)]a_i(j+k) + a_i(-j)[a_r(n),a_i(j+k)] \Big\} \Psi(\varepsilon j)$$

$$= \frac{1}{2} \sum_{j\in\mathbb{Z}} \sum_{i=1}^{m} \Big\{ (a_r|a_i)n\delta_{n,j}\, a_i(j+k) + (a_r|a_i)n\delta_{n+j+k,0} a_i(-j) \Big\} \Psi(\varepsilon j), \text{ by (1.1.4)}$$

$$= \frac{1}{2}\, n\, a_r(n+k)\, \Psi(\varepsilon n) + \frac{1}{2}\, n\, a_r(n+k)\, \Psi(\varepsilon(-n-k))$$

$$= n\, a_r(n+k), \text{ by letting } \varepsilon \to 0.$$

$$\text{Next } [L_k(\varepsilon), L_\ell] = \frac{1}{2} \sum_{j\in\mathbb{Z}} \sum_{i=1}^{m} [a_i(-j)a_i(j+k), L_\ell]\, \Psi(\varepsilon j)$$

$$= \frac{1}{2} \sum_{j\in\mathbb{Z}} \sum_{i=1}^{m} (-j)a_i(\ell-j)a_i(j+k)\, \Psi(\varepsilon j)$$

$$+ \frac{1}{2} \sum_{j\in\mathbb{Z}} \sum_{i=1}^{m} (j+k)a_i(-j)a_i(\ell+j+k)\, \Psi(\varepsilon j).$$

Now we must use (2.2.2) to put the normal ordering back in so that we can identify the operator $L_{k+\ell}$ in this sum. The terms in the first sum which are not in normal order are those for which $j < \frac{\ell-k}{2}$ and the terms in the second sum which are not in normal order are those satisfying $j < -\frac{(\ell+k)}{2}$. Thus

$$(2.2.3) \qquad [L_k(\varepsilon), L_\ell] = \frac{1}{2} \sum_{j\in\mathbb{Z}} \sum_{i=1}^{m} (-j)\, :a_i(\ell-j)a_i(j+k): \Psi(\varepsilon j)$$

$$+ \frac{1}{2} \sum_{j\in\mathbb{Z}} \sum_{i=1}^{m} (j+k)\, :a_i(-j)a_i(\ell+j+k): \Psi(\varepsilon j)$$

$$+ \frac{m}{2}\, \delta_{k,-\ell} \Big\{ \sum_{j<-k} (j)(j+k)\, \Psi(\varepsilon j) - \sum_{j<0} (j)(j+k)\, \Psi(\varepsilon j) \Big\}, \text{ since } \sum_{i=1}^{m} (a_i|a_i) = m.$$

Now replacing j by $j+\ell$ in the first sum and taking the limit $\varepsilon\to 0$ the first two sums combine to give $(k-\ell)L_{k+\ell}$. To simplify the sum in braces note that

when $k>0$ it becomes $-\sum_{j=-1}^{k} j(j+k)\,\Psi(\varepsilon j)$. However,

$$-\frac{1}{2}\sum_{j=-1}^{-k} j(j+k) = \frac{(k^3-k)}{12} \quad \text{since} \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{and} \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

When $k<0$ the sum becomes $\sum_{j=0}^{-k-1} j(j+k)\,\Psi(\varepsilon j)$ which is the same as

$\sum_{j=1}^{-k} j(j+k)\,\Psi(\varepsilon j)$ since $j(j+k)=0$ when $j\in(0,-k)$. But again

$\frac{1}{2}\sum_{j=1}^{-k} j(j+k) = \frac{(k^3-k)}{12}$. Thus letting $\varepsilon\to 0$ the expression in braces in (2.2.3)

becomes $\frac{k^3-k}{12}$ when $k\neq 0$. If $k=0$ the last sum in (2.2.3) is clearly zero and

$\frac{k^3-k}{12} = 0$. Thus we have shown that $[L_k,L_\ell] = (k-\ell)L_{k-\ell} + m\delta_{k,-\ell}\frac{(k^3-k)}{12}$ and

hence the proof is complete. ∎

§2.3 Oscillator Representations of $\mathcal{V}ir$ over the Hyperbolic Lattice Λ

Recall from §1.5 that $\Lambda := \mathbb{Z}\delta \oplus \mathbb{Z}\mu$ is the even unimodular indefinite lattice whose matrix with respect to the form $(\cdot|\cdot)$ on Λ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Λ is called a *Hyperbolic lattice*. Let $\mathfrak{p} := \mathbb{C}\otimes_{\mathbb{Z}}\Lambda$ be the complexification of Λ, let Θ be the symmetric algebra of the $\mathbb{C}$-space $\coprod_{n>0} \mathfrak{p}(-n)$ and finally, let $V_\Lambda(\lambda) = \mathbb{C}e^{\lambda}\otimes_{\mathbb{C}}\Theta$, $\lambda\in\mathfrak{p}$, be the canonical representation of $\mathfrak{a}(\Lambda)$. Introduce the following elements of $\mathfrak{p}$:

$$\text{(2.3.1)} \qquad \begin{cases} \alpha_1 := \delta/2 + \mu \\ \alpha_2 := i\left(\delta/2 - \mu\right),\ i^2 := -1 \end{cases}$$

It is easy to check that, with respect to the unique symmetric bilinear form $(\cdot|\cdot)$ on $\mathfrak{p}$ extending the above form, the set $\{\alpha_1,\alpha_2\}$ is an orthonormal basis for $\mathfrak{p}$.

Thus, by proposition 2.2.1, for $\lambda\in\mathfrak{p}$ the Fock space $V_\Lambda(\lambda)$ affords a representation of $\mathcal{V}ir$ under the assignment

$$\text{(2.3.2)} \qquad \begin{cases} d_k \longmapsto H_k \\ \mathfrak{z} \longmapsto 2I \end{cases}$$

where $H_k := \frac{1}{2}\sum_{j\in\mathbb{Z}}\sum_{i=1}^{2} :\alpha_i(-j)\alpha_i(j+k):$.

Notation: Let $L_1 := \mathbb{Z}\alpha_1$ and $L_2 := \mathbb{Z}\alpha_2$ where α_1 and α_2 are as above and put $\mathfrak{p}_1 := \mathbb{C}\otimes_{\mathbb{Z}} L_1$ and $\mathfrak{p}_2 := \mathbb{C}\otimes_{\mathbb{Z}} L_2$. Then $\mathfrak{p} \cong \mathfrak{p}_1 \oplus \mathfrak{p}_2$. For $\lambda\in\Lambda$ write $\lambda = \lambda_1+\lambda_2$ where $\lambda_1\in\mathfrak{p}_1$ and $\lambda_2\in\mathfrak{p}_2$. Then $V_{L_1}(\lambda_1)$ and $V_{L_2}(\lambda_2)$ each afford representations of $\mathcal{V}ir$ under the assignments

$$\text{(2.3.3)} \qquad \begin{cases} d_k \longmapsto L_k^{(i)} \\ \mathfrak{z} \longmapsto I. \end{cases}$$

where $L_k^{(i)} := \frac{1}{2}\sum_{j\in\mathbb{Z}} :\alpha_i(-j)\alpha_i(j+k):$, $i=1,2$.

Proposition 2.3.1: Under the above notation, we have (as modules over $\mathcal{V}ir$)

$$\text{(2.3.4)} \qquad V_\Lambda(\lambda) \cong V_{L_1}(\lambda_1) \otimes_{\mathbb{C}} V_{L_2}(\lambda_2).$$

Proof: Recall $V_\Lambda(\lambda) = \mathbb{C}e^\lambda\otimes_{\mathbb{C}} S(\mathfrak{a}(\Lambda)_-) = \mathbb{C}e^\lambda\otimes_{\mathbb{C}} S\left(\coprod_{n>0}\mathfrak{p}(-n)\right)$, $\mathfrak{p}(n) \cong \mathfrak{p} = \mathbb{C}\delta \oplus \mathbb{C}\mu$. Let $\Theta_1 := S\left(\coprod_{n>0}\mathbb{C}\alpha_1(-n)\right)$ and $\Theta_2 := S\left(\coprod_{n>0}\mathbb{C}\alpha_2(-n)\right)$. Then $\Theta = \Theta_1\Theta_2$ where $\Theta_1\Theta_2$ denotes the smallest subalgebra of Θ containing both Θ_1 and Θ_2. Moreover, $V_{L_1}(\lambda_1) = \mathbb{C}e^{\lambda_1}\otimes_{\mathbb{C}}\Theta_1$ and $V_{L_2}(\lambda_2) = \mathbb{C}e^{\lambda_2}\otimes_{\mathbb{C}}\Theta_2$. Now let $\phi : V_\Lambda(\lambda) \longrightarrow V_{L_1}(\lambda_1) \otimes_{\mathbb{C}} V_{L_2}(\lambda_2)$ be the unique linear map satisfying $x := e^\lambda\otimes \alpha_1(-n)\alpha_2(-m) \longmapsto (e^{\lambda_1}\otimes \alpha_1(-n)) \otimes (e^{\lambda_2}\otimes \alpha_2(-m)) =: x_1\otimes x_2$ where $n\in\mathbb{Z}_+^s$, $m\in\mathbb{Z}_+^r$, $r,s>0$. Clearly ϕ is a linear isomorphism since it takes a basis to a basis. We need only show that $\phi(H_k\cdot x) = \left(L_k^{(1)}\otimes I\right)+\left(I\otimes L_k^{(2)}\right)(x_1\otimes x_2)$.

Indeed, $\phi(H_k\cdot x) = \phi\left\{\frac{1}{2}\sum_{j\in\mathbb{Z}} :\alpha_1(-j)\alpha_1(j+k): (e^\lambda\otimes \alpha_1(-n)\alpha_2(-m))\right.$

$$\left. + \frac{1}{2}\sum_{j\in\mathbb{Z}} :\alpha_2(-j)\alpha_2(j+k): (e^\lambda\otimes \alpha_1(-n)\alpha_2(-m))\right\}$$

$$= \phi \left\{ \phi^{-1} \left[L_k^{(1)}(e^{\lambda_1} \otimes \alpha_1(-n)) \otimes (e^{\lambda_2} \otimes \alpha_2(-m)) \right] \right.$$

$$\left. + \phi^{-1} \left[(e^{\lambda_1} \otimes \alpha_1(-n)) \otimes (L_k^{(2)}(e^{\lambda_2} \otimes \alpha_2(-m)) \right] \right\}$$

$$= \left[L_k^{(1)} x_1 \otimes x_2 \right] + \left[x_1 \otimes L_k^{(2)} x_2 \right]$$

$$= \left[L_k^{(1)} \otimes I \right] + \left[I \otimes L_k^{(2)} \right] (x_1 \otimes x_2). \quad \blacksquare$$

As outlined in §2.1 the structure of the modules $V_{L_i}(\lambda_i)$, i=1,2 is well-understood. However, as pointed out in the introduction, for our purposes it will not be useful to understand $V_\Lambda(\lambda)$ in terms of the components $V_{L_i}(\lambda_i)$, i=1,2 occurring in (2.3.4). To this end, although it was useful to work in the $\{\alpha_1,\alpha_2\}$-basis to produce the representation (2.3.2) and to obtain proposition 2.3.1, from here on we will work in the $\{\delta,\mu\}$-basis. Our first goal will be to express the module $V_\Lambda(\lambda)$, $\lambda\in\Lambda$ and the operators H_k, $k\in\mathbb{Z}$ in this basis.

From §1.4 we know that for $\lambda\in\mathfrak{p}$, $V_\Lambda(\lambda)$ is constructed in the following way. For each $n\in\mathbb{Z}$ we let $\mathfrak{p}(n)$ be an isomorphic copy of $\mathfrak{p}$ (as a $\mathbb{C}$-space) under the correspondence $a(n) \longleftrightarrow a$, $a\in\mathfrak{p}$.

Since $\mathfrak{p} = \mathbb{C}\delta\oplus\mathbb{C}\mu$, for every $n\in\mathbb{Z}$, we have $\mathfrak{p}(n) = \mathbb{C}\delta(n)\oplus\mathbb{C}\mu(n)$. Now introduce the subalgebras $M := S\left(\coprod_{n>0} \mathbb{C}\mu(-n) \right)$ and $D := S\left(\coprod_{n>0} \mathbb{C}\delta(-n) \right)$ of Θ. Using a basis argument, it is easy to see that $\Theta = MD$ and hence

$$W(\lambda) := V_\Lambda(\lambda) = \mathbb{C}e^\lambda \otimes_{\mathbb{C}} \Theta = \mathbb{C}e^\lambda \otimes_{\mathbb{C}} MD.$$

As for the operators H_k, $k\in\mathbb{Z}$ we have

Proposition 2.3.2: For every $n\in\mathbb{Z}$

(i) $H_n = \frac{1}{2} \sum_{j\in\mathbb{Z}} \left(:\mu(-j)\delta(j+n): + :\delta(-j)\mu(j+n): \right)$

(ii) $H_n = H_n^- + H_n^+$ where

$$H_n^- := \frac{\varepsilon}{2}\mu(n/2)\delta(n/2) + \sum_{j>-n/2} \mu(-j)\delta(j+n) ,$$

$$H_n^+ := \frac{\varepsilon}{2}\delta(n/2)\mu(n/2) + \sum_{j>-n/2} \delta(-j)\mu(j+n)$$

and $\varepsilon := \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} . \end{cases}$

Proof: For any $j\in\mathbb{Z}$, $:\alpha_1(-j)\alpha_1(j+n): + :\alpha_2(-j)\alpha_2(j+n):$

$$= \begin{cases} \alpha_1(-j)\alpha_1(j+n) + \alpha_2(-j)\alpha_2(j+n), & -j\le j+n \\ \\ \alpha_1(j+n)\alpha_1(-j) + \alpha_2(j+n)\alpha_2(-j), & -j>j+n. \end{cases}$$

Thus, for (i), it suffices to show that

$\alpha_1(-j)\alpha_1(j+n) + \alpha_2(-j)\alpha_2(j+n) = \mu(-j)\delta(j+n) + \delta(-j)\mu(j+n)$

and $\alpha_1(j+n)\alpha_1(-j) + \alpha_2(j+n)\alpha_2(-j) = \mu(j+n)\delta(-j) + \delta(j+n)\mu(-j).$

We show only the first of these since the second is similar.

$$\alpha_1(-j)\alpha_1(j+n) + \alpha_2(-j)\alpha_2(j+n)$$

$$= \left(\frac{\delta(-j)}{2} + \mu(-j)\right)\left(\frac{\delta(j+n)}{2} + \mu(j+n)\right) - \left(\frac{\delta(-j)}{2} - \mu(-j)\right)\left(\frac{\delta(j+n)}{2} - \mu(j+n)\right)$$

$$= \frac{1}{4}\delta(-j)\delta(j+n) + \frac{1}{2}\delta(-j)\mu(j+n) + \frac{1}{2}\mu(-j)\delta(j+n) + \mu(-j)\mu(j+n)$$

$$- \frac{1}{4}\delta(-j)\delta(j+n) + \frac{1}{2}\delta(-j)\mu(j+n) + \frac{1}{2}\mu(-j)\delta(j+n) - \mu(-j)\mu(j+n)$$

$= \delta(-j)\mu(j+n) + \mu(-j)\delta(j+n)$. This proves (i).

For (ii), we first use (i) and then remove normal ordering:

$$H_n = \frac{1}{2}\sum_{j\in\mathbb{Z}} \left(:\mu(-j)\delta(j+n): + :\delta(-j)\mu(j+n):\right)$$

$$= \frac{1}{2}\sum_{-j\le j+n} \left(\mu(-j)\delta(j+n) + \delta(-j)\mu(j+n)\right) + \frac{1}{2}\sum_{-j>j+n} \left(\delta(j+n)\mu(-j) + \mu(j+n)\delta(-j)\right)$$

$$= \sum_{j>-n/2} \left(\mu(-j)\delta(j+n) + \delta(-j)\mu(j+n)\right) + \frac{1}{2}\sum_{j>-n/2} \left(\delta(-j)\mu(j+n) + \mu(-j)\delta(j+n)\right)$$

$$+ \frac{\varepsilon}{2}\left(\mu(n/2)\delta(n/2) + \delta(n/2)\mu(n/2)\right)$$

where we have split the first sum into $j = -n/2$, $j>-n/2$ and replaced j by $-j-n$ in the second sum. Regrouping we have

$$H_n = \sum_{j>-n/2} \mu(-j)\delta(j+n) + \frac{\varepsilon}{2}\mu(n/2)\delta(n/2) + \sum_{j>-n/2} \delta(-j)\mu(j+n) + \frac{\varepsilon}{2}\delta(n/2)\mu(n/2).\blacksquare$$

Remarks: (i) An alternate way of expressing $H_n^{\pm}$ is

$$H_n^- = \sum_{\substack{i<j \\ i+j=n}} \mu(i)\delta(j) + \frac{\varepsilon}{2}\mu(n/2)\delta(n/2) \quad \text{and}$$

$$H_n^+ = \sum_{\substack{i<j \\ i+j=n}} \delta(i)\mu(j) + \frac{\varepsilon}{2}\delta(n/2)\mu(n/2).$$

(ii) Our original definition of the L_k in (2.2.1) was constructed from a choice of an orthonormal basis for $\ell = \mathbf{C}e_{\mathbf{Z}}L$. It is possible to construct the L_k using a dual pair of bases and immediately obtain the form of the H_k in proposition 2.3.2 (i).

§2.4 A Filtration of W(λ) and a Decomposition of W(0)

The purpose of this section is to show that W(0) can be decomposed into a direct sum of submodules $\overline{W_k(0)}$, $k\in\mathbb{Z}$. Moreover, when $0\neq\lambda\in\mathbb{Z}\delta$, W(λ) admits a filtration of submodules $W^k(\lambda)$, $k\in\mathbb{Z}$ whose subquotients $W_k(\lambda) := W^k(\lambda)/W^{k-1}(\lambda)$ are isomorphic to $\overline{W_k(0)}$ as modules over $\mathcal{V}ir$.

Note that $M = S\left(\coprod_{n>0} \mathbb{C}\mu(-n)\right)$ has the following $\mathbb{C}$-basis:

$\{\ \mu(-\mathbf{n}) := \mu(-n_1)\cdots\mu(-n_s) : \mathbf{n}:=(n_1,\ldots,n_s)\in\mathbb{Z}_+^s,\ s\geq 0\ \}$.

For $\mathbf{n}\in\mathbb{Z}_+^s$ we say that $\mu(-\mathbf{n})$ has *length* s and we write $\ell(\mu(-\mathbf{n})) = s$. Now introduce the following subspaces of M:

Let $\Omega_j := (0)$, if $j<0$

$\Omega_0 := \mathbb{C}$

$\Omega_j :=$ $\mathbb{C}$-span of all monomials in M of length j, $j>0$

$\Omega_{\leq j} := \coprod_{k\leq j} \Omega_k$.

Then $M = \coprod_{j\geq 0} \Omega_j$.

Similarly, a $\mathbb{C}$-basis for $D = S\left(\coprod_{m>0} \mathbb{C}\delta(-m)\right)$ is given by

$\{\ \delta(-\mathbf{m}) := \delta(-m_1)\cdots\delta(-m_r) : \mathbf{m} = (m_1,\ldots,m_r)\in\mathbb{Z}_+^r\ ,\ r\geq 0\ \}$.

As above, we say that $\delta(-\mathbf{m})$ has *length* r if $\mathbf{m}\in\mathbb{Z}_+^r$. Define

$\Delta_j := (0)$ if $j<0$

$\Delta_0 := \mathbb{C}$

$\Delta_j :=$ $\mathbb{C}$-span of all monomials in D of length j.

Then $D = \coprod_{j\geq 0} \Delta_j$.

Next introduce $\Theta^k := \coprod_{j\geq 0}\left(\Omega_{\leq(j+k)}\Delta_j\right) \subset \Theta$. Then, as $\mathbb{C}$-spaces, we have

$$\Theta_k := \Theta^k/\Theta^{k-1} \cong \coprod_{j\geq 0}\left(\Omega_{(j+k)}\Delta_j\right). \tag{2.4.1}$$

For $k\in\mathbb{Z}$ and $\lambda\in\mathbb{Z}\delta$ we define $W^k(\lambda) := \mathbb{C}e^{\lambda}\otimes_{\mathbb{C}}\Theta^k$, $W_k(\lambda) := W^k(\lambda)/W^{k-1}(\lambda)$ and finally $\overline{W_k(0)} := \mathbb{C}e^0\otimes_{\mathbb{C}}\coprod_{j\geq 0}\left(\Omega_{(j+k)}\Delta_j\right)$. Then we have

Proposition 2.4.1: (i) For any $\lambda\in\mathbb{Z}\delta$ $\{W^k(\lambda) : k\in\mathbb{Z}\}$ is a filtration of submodules of $W(\lambda)$ satisfying $W^k(\lambda) \subset W^{k+1}(\lambda)$, $k\in\mathbb{Z}$.
(ii) $\{\overline{W_k(0)} : k\in\mathbb{Z}\}$ is a collection of submodules of $W(0)$ satisfying $W(0) = \coprod_{k\in\mathbb{Z}} \overline{W_k(0)}$.

Proof: It is clear from the definitions of $W^k(\lambda)$, $W_k(\lambda)$ and $W(\lambda)$ that $W^k(\lambda) \subset W^{k+1}(\lambda)$, $k\in\mathbb{Z}$ and $W(0) = \coprod_{k\in\mathbb{Z}} \overline{W_k(0)}$. Thus, to prove part (i) of the proposition, it suffices to show that $W^k(\lambda)$ is a submodule of $W(\lambda)$ for any $k\in\mathbb{Z}$ and any $\lambda\in\mathbb{Z}\delta$. For part (ii), we need to show that $\overline{W_k(0)}$ is a submodule of $W(0)$.

Let $x = e^{\lambda}\otimes\mu(-n)\delta(-m) \in W^k(\lambda)$. We can assume $\mu(-n) \in \Omega_{(j+k)}$ where $\delta(-m) \in \Delta_j$ for some $j\in\mathbb{Z}$. Let $n\in\mathbb{Z}$ be arbitrary. It suffices to show that $H_n^{\pm}(x) \in W^k(\lambda)$. Indeed, let $H_n^-(i,j) := \mu(i)\delta(j)$ and $H_n^+(i,j) := \delta(i)\mu(j)$. Then we have

$$H_n^- = \sum_{\substack{i<j\\ i+j=n}} \mu(i)\delta(j) + \frac{\varepsilon}{2}\,\mu(n/2)\delta(n/2) = \sum_{(i,j)\in I} H_n^-(i,j) + \frac{\varepsilon}{2}\,\mu(n/2)\delta(n/2)$$

where $I := \{t:=(i,j) : i<j \text{ and } i+j=n\}$ and similarly for H_n^+.

Consider all *types* of pairs (i,j) according to whether i and j are positive, negative or zero. There are nine such types. However, if we restrict ourselves to those for which i<j then we are left with only six types since (+,0), (+,-) and (0,-) would not be admissible.

It is important for us to first focus on the pair-type, instead of the actual pair entries, since the structure of our modules $W^k(\lambda)$ is determined by certain monomial-lengths and it is the pair-type which determines the effect that $H_n^{\pm}(i,j)$ has on the monomial-length.

When we compute $H_n^{\pm}(x) = \sum_{(i,j)\in I} H_n^{\pm}(i,j)(x)$ we obtain a sum of monomials

$\sum_{t\in I} c_t \; e^{\lambda}\otimes_{\mathbb{C}}\mu(-n_t)\delta(-m_t)$, one for each admissible pair t=(i,j). However, all but a finite number of the c_t will be zero. That is, only finitely many admissible pairs (i,j) will contribute in a nontrivial way. For example, consider $H_3(e^{\lambda}\otimes\mu(-4)\delta(1))$. Although there are infinitely many admissible pairs (i,j) with i<j and i+j=3, there is only one such pair which does not produce zero. Namely (-1,4) and it is of type (-,+). Note $H_3(e^{\lambda}\otimes\mu(-4)\delta(-1)) = H_3^-(-1,4)(e^{\lambda}\otimes\mu(-4)\delta(-1)) = 4(e^{\lambda}\otimes\mu(-1)\delta(-1))$. Since we are interested in checking that $W^k(\lambda)$ is an invariant subspace, the table below records only the effect in the above sense. If for a given pair-type all admissible pairs produce zero, then we record this with the phrase "dies". Finally, if the μ or δ length is increased (resp. decreased) by one then we indicate this in the table by writing "↑ by 1" (resp. "↓ by 1"). The last two rows of the table differ in the cases $\lambda=0$ and $\lambda\neq 0$. We record the effect in both cases.

Pair Type of (i,j)	Effect of $H_n^-(i,j)$ on $(\ell(\mu(-n)),\ell(\delta(-m)))$	Effect of $H_n^+(i,j)$ on $(\ell(\mu(-n)),\ell(\delta(-m)))$
$(+,+)$	($\downarrow$ by 1 , $\downarrow$ by 1)	($\downarrow$ by 1 , $\downarrow$ by 1)
$(-,-)$	($\uparrow$ by 1 , $\uparrow$ by 1)	($\uparrow$ by 1 , $\uparrow$ by 1)
$(0,0)$	dies	dies
$(-,+)$	(none , none)	(none , none)
$(0,+)$	($\downarrow$ by 1, none) if $\lambda\neq 0$ dies if $\lambda=0$	dies
$(-,0)$	dies	(none, $\uparrow$ by 1) if $\lambda\neq 0$ dies if $\lambda=0$

Table 1

Now suppose $\lambda\in\mathbb{Z}\delta$ is arbitrary. Then including all five possibilities from the table we have:

$$\begin{aligned}\mathfrak{U}(\mathcal{V}ir)\cdot x &\subset \mathbb{C}e^{\lambda}\otimes_{\mathbb{C}}\left\{\left(\Omega_{\leq(j+k-1)}\Delta_{j-1}\right)\oplus\left(\Omega_{\leq(j+k+1)}\Delta_{j+1}\right)\oplus\left(\Omega_{\leq(j+k)}\Delta_{j}\right)\oplus\right.\\ &\qquad\left.\left(\Omega_{\leq(j+k-1)}\Delta_{j}\right)\oplus\left(\Omega_{\leq(j+k)}\Delta_{j+1}\right)\right\}\\ &\subset \mathbb{C}e^{\lambda}\otimes_{\mathbb{C}}\left\{\left(\Omega_{\leq(j+k-1)}\Delta_{j-1}\right)\oplus\left(\Omega_{\leq(j+k)}\Delta_{j}\right)\oplus\left(\Omega_{\leq(j+k+1)}\Delta_{j+1}\right)\right\}\\ &\subset W^{k}(\lambda).\end{aligned}$$

Finally if $\lambda=0$ then we have

$$\mathfrak{U}(\mathcal{V}ir)\cdot x \subset \mathbb{C}e^{0}\otimes_{\mathbb{C}}\left\{\left(\Omega_{j+k-1}\Delta_{j-1}\right)\oplus\left(\Omega_{j+k+1}\Delta_{j+1}\right)\oplus\left(\Omega_{j+k}\Delta_{j}\right)\right\}\subset\overline{W_{k}(0)}.$$

This proves (ii). ∎

Proposition 2.4.2: Fix any $k\in\mathbb{Z}$ and $\lambda\in\mathbb{Z}\delta$ and consider the submodule $\overline{W_k(0)}$ of $W(0)$ and the subquotient $W_k(\lambda) = W^k(\lambda)/W^{k-1}(\lambda)$ of $W(\lambda)$. Then, as modules over $\mathcal{V}ir$, we have $\overline{W_k(0)} \cong W_k(\lambda)$.

Proof: Let $w = \sum_{\ell=1}^{r} c_\ell(e^0\otimes x_\ell) \in W_k(0)$, $c_\ell\in\mathbb{C}$, $x_\ell \in \left(\coprod_{j\geq 0}\Omega_{j+k}\Delta_j\right)$. Now define a map $\overline{W_k(0)} \longrightarrow W^k(\lambda)$ by $w \longmapsto w_\lambda$ where $w_\lambda := \sum_{\ell=1}^{r} c_\ell(e^\lambda\otimes x_\ell)$.

Let $\phi : \overline{W_k(0)} \longrightarrow W_k(\lambda)$ be defined by $\phi(w) := \overline{w_\lambda} := w_\lambda + W^{k-1}(\lambda)$. It is clear that ϕ is a linear isomorphism. To see that ϕ is a $\mathcal{V}ir$-module map it suffices to check that $\phi(H_k\cdot w) = H_k\cdot\phi(w)$, $w \in \overline{W_k(0)}$ or, equivalently,

$$(H_k\cdot w)_\lambda = H_k\cdot w_\lambda \mod W^{k-1}(\lambda).$$

For this we observe from Table 1 that in the calculation of

$$H_k\cdot w = \sum_{\ell=1}^{r} c_\ell \left\{ \sum_{(i,j)\in I} \left(H_k^+(i,j) + H_k^-(i,j)\right)\right\}\cdot(e^0\otimes x_\ell) \quad \text{in } \overline{W_k(0)} \text{ and}$$

$$H_k\cdot w_\lambda = \sum_{\ell=1}^{r} c_\ell\left\{ \sum_{(i,j)\in I} \left(H_k^+(i,j) + H_k^-(i,j)\right)\right\}\cdot(e^\lambda\otimes x_\ell) \quad \text{in } W_k(\lambda)$$

the action only differs when at least one of i or j is zero and, in these cases, $H_k^{\pm}(i,j)\cdot(e^0\otimes x) = 0$ and $H_k^{\pm}(i,j)\cdot(e^\lambda\otimes x) = 0 \mod W^{k-1}(\lambda)$. This proves the proposition. ∎

Corollary 2.4.3: For every $k\in\mathbb{Z}$ and $\lambda\in\mathbb{Z}\delta$

$$0 \longrightarrow W^{k-1}(\lambda) \longrightarrow W^k(\lambda) \longrightarrow \overline{W_k(0)} \longrightarrow 0$$

is a short exact sequence of $\mathcal{V}ir$-modules.

§2.5 A Positive Definite Hermitian Form $\langle\cdot|\cdot\rangle$ on $W(\lambda)$

In this section we construct a positive definite Hermitian form $\langle\cdot|\cdot\rangle$ on $W(\lambda)$, $\lambda\in\mathbb{Z}\delta$ and we find both an orthogonal basis for $W(\lambda)$ and an explicit formula for the square norm of any element in this basis with respect to $\langle\cdot|\cdot\rangle$.

We begin by extending the original $\mathbb{Z}$-bilinear form $(\cdot|\cdot)$ on Λ to a Hermitian form on $\Theta = S\left(\coprod_{n>0} p(-n)\right)$, also denoted $(\cdot|\cdot)$, in such a way that for $a_i, b_i \in \{\delta,\mu\}$

$$(2.5.1)\qquad \begin{cases} (a_1(-n_1)\cdots a_s(-n_s)\,|\,b_1(-m_1)\cdots b_r(-m_r)) \\ = \delta_{r,s} \sum_{\sigma\in\mathfrak{S}_r} \prod_{k=1}^{r} n_k\, \delta_{n_k,m_{\sigma(k)}}\, (a_k|b_{\sigma(k)}) \end{cases}$$

where $\delta_{x,y}$, $x,y\in\mathbb{Z}$ denotes the usual Kronecker-delta and $\mathfrak{S}_r$ denotes the symmetric group on r symbols. We treat $(\cdot|\cdot)$ linear in the first component and anti-linear in the second component. For $\mathbf{a} := (a_1,\ldots,a_s)$, $\mathbf{b} := (b_1,\ldots,b_r)$, $\mathbf{n} := (n_1,\ldots,n_s)$ and $\mathbf{m} := (m_1,\ldots,m_r)$ we let $\mathbf{a}(-\mathbf{n}) := a_1(-n_1)\cdots a_s(-n_s)$ and $\mathbf{b}(-\mathbf{m}) := b_1(-m_1)\cdots b_r(-m_r) \in \Theta$. If $\mathbf{a} = (\mu,\ldots,\mu)$ and $\mathbf{b} = (\delta,\ldots,\delta)$ then we denote $\mathbf{a}(-\mathbf{n})$ by $\mu(-\mathbf{n})$ and $\mathbf{b}(-\mathbf{m})$ by $\delta(-\mathbf{m})$.

In the above definitions, $\mathbf{n} = (n_1,\ldots,n_s)$ is an ordered s-tuple. If we want to simply regard it as a set of s elements with no specific ordering we will write $(\mathbf{n})$. Now, for any $\mathbf{a},\mathbf{b}$ as above, (2.5.1) gives

$$(\mathbf{n}) \neq (\mathbf{m}) \;\Rightarrow\; (\mathbf{a}(-\mathbf{n})|\mathbf{b}(-\mathbf{m})) = 0$$

We make some additional useful definitions. For $\mathbf{n},\mathbf{m} \in \mathbb{Z}_+^r$ define

$$\delta_{n,m} := \begin{cases} 1\ , & (n)=(m) \\ 0\ , & \text{otherwise.} \end{cases}$$

$\pi(n) := n_1 \cdots n_r = \prod_{i=1}^{r} n_i$ and $c(n,m) := \text{card}(\sigma \in \mathfrak{S}_r : \sigma(n)=m)$.

Proposition 2.5.1: If $n = (\underbrace{n_1,\ldots,n_1}_{a_1\text{-times}}, \ldots, \underbrace{n_p,\ldots,n_p}_{a_p\text{-times}}, \ldots, \underbrace{n_v,\ldots,n_v}_{a_v\text{-times}}) \in Z_+^s$,

ie. $\sum_{i=1}^{v} a_i = s$ and $m \in Z_+^s$ is arbitrary then

(i) $c(n,m) = \delta_{n,m}\, c(n,n)$

(ii) $c(n) := c(n,n) = a_1! \cdots a_p! \cdots a_v!$

(iii) $(\mu(-n)|\delta(-m)) = \delta_{n,m}\, c(n)\pi(n)$.

Proof: (i) It is clear that if $(n)\neq(m)$ then $c(n,m)=0$. There is still one more thing to check since $(n)=(m)$ does not, of course, imply $n=m$. However, if $(n)=(m)$ then one simply observes that the number of elements $\sigma \in \mathfrak{S}_s$ satisfying $\sigma(n)=m$ is equal to the number of elements $\sigma \in \mathfrak{S}_s$ satisfying $\sigma(n)=n$.

(ii) There are $a_1!$ permutations of (n) which fix $n_2, n_3, \ldots, n_v$ and permute the n_1's. For each of these there are $a_2!$ ways of permuting (n) which fix $n_1, n_3, \ldots, n_v$ and permute the n_2's. Since this is true for each $1 \le i \le v$, (ii) is now clear. Finally for (iii), note that by (2.5.1)

$$\begin{aligned} (\mu(-n)|\delta(-m)) &= \sum_{\sigma \in \mathfrak{S}_s} \prod_{i=1}^{s} n_k\, \delta_{n_k, m_{\sigma(k)}} \quad (\mu|\delta) \\ &= \delta_{n,m}\, c(n,m)\pi(n) \\ &= \delta_{n,m}\, c(n)\pi(n), \text{ by part (i).} \end{aligned}$$

■

Proposition 2.5.2: Let $n \in Z_+^s$, $n' \in Z_+^{s'}$, $m \in Z_+^r$ and $m' \in Z_+^{r'}$. Then

$(\mu(-n)\delta(-m)|\mu(-n')\delta(-m')) = (\mu(-n)|\delta(-m'))\,(\mu(-n')|\delta(-m))$

Proof: Note that if $r \neq s'$ or $r' \neq s$ then by (2.5.1) it is clear that every summand in the left side has a factor of the form $(\delta|\delta) = 0 = (\mu|\mu)$ and hence the left side is zero. In this case, the previous proposition guarantees that the right side is zero. Thus we assume $r=s'$ and $r'=s$.

Now $LS = (\mu(-n)\delta(-m)|\mu(-n')\delta(-m'))$

$$= \sum_{\sigma \in \mathfrak{S}_{r+s}} \prod_{k=1}^{r+s} p_k \, \delta_{p_k, q_{\sigma(k)}} (a_k | b_{\sigma(k)})$$

where $p_k := \begin{cases} n_k & , 1 \le k \le s \\ m_{k-s} & , s < k \le s+r \end{cases}$, $q_k := \begin{cases} n'_k & , 1 \le k \le r \quad (\text{since } r=s') \\ m'_{k-r} & , r < k \le s+r \quad (\text{since } s=r') \end{cases}$

$a_k := \begin{cases} \mu & , 1 \le k \le s \\ \delta & , s < k \le s+r \end{cases}$, $b_k := \begin{cases} \mu & , 1 \le k \le r \\ \delta & , r < k \le r+s. \end{cases}$

$$\text{Thus } LS = \sum_{\sigma \in \tilde{\mathfrak{S}}_{r+s}} \left(\prod_{k=1}^{s} n_k \, \delta_{n_k, m'_{\sigma(k)-r}} (\mu|\delta) \right) \left(\prod_{k=s+1}^{s+r} m_{k-s} \, \delta_{m_{k-s}, n'_{\sigma(k)}} (\delta|\mu) \right)$$

where $\tilde{\mathfrak{S}}_{r+s}$ is the collection of all elements $\sigma \in \mathfrak{S}_{r+s}$ satisfying $r+1 \le \sigma(k) \le r+s$, $1 \le k \le s$ and $1 \le \sigma(k) \le r$, $s+1 \le k \le s+r$. Other summands in the left side are zero because they include a factor of the form $(\delta|\delta)=0$ or $(\mu|\mu)=0$. Next we replace k by $k+s$ in the second factor and get:

$$LS = \sum_{\sigma\in\tilde{\mathfrak{S}}_{r+s}} \left(\prod_{k=1}^{s} n_k \, \delta_{n_k, m'_{\sigma(k)-r}} \right) \left(\prod_{k=1}^{r} m_k \, \delta_{m_k, n'_{\sigma(k+s)}} \right).$$

Now, for any $\sigma\in\tilde{\mathfrak{S}}_{r+s}$, the elements τ_1 and τ_2 defined by

$$\begin{cases} \tau_1(k) := \sigma(k)-r\,, & 1\le k\le s \\ \tau_2(k) := \sigma(k+s)\,, & 1\le k\le r \end{cases}$$

belong to $\mathfrak{S}_s$ and $\mathfrak{S}_r$ respectively. Conversely, given any $\tau_1\in\tilde{\mathfrak{S}}_s$ and $\tau_2\in\mathfrak{S}_r$ the above two equations define a unique $\sigma\in\mathfrak{S}_{r+s}$. Thus we have

$$LS = \left(\sum_{\tau_1\in\mathfrak{S}_s} \prod_{k=1}^{s} n_k \, \delta_{n_k, m'_{\tau_1(k)}} \right) \left(\sum_{\tau_2\in\mathfrak{S}_r} \prod_{k=1}^{r} m_k \, \delta_{m_k, n'_{\tau_2(k)}} \right)$$

$$= (\mu(-\mathbf{n})|\delta(-\mathbf{m}')) \; (\delta(-\mathbf{m})|\mu(-\mathbf{n}')). \qquad \blacksquare$$

Definitions: Define $\vartheta : \Theta \longrightarrow \Theta$ as the unique linear map satisfying $\vartheta(\mu(-\mathbf{n})\delta(-\mathbf{m})) := \mu(-\mathbf{m})\delta(-\mathbf{n})$ where $\mathbf{n}\in\mathbb{Z}_+^s$, $\mathbf{m}\in\mathbb{Z}_+^r$ and $r,s\ge 0$. Then ϑ is an involution and identifying $\coprod_{j\ge 0}\left(\Omega_{(j+k)}\Delta_j\right)$ with Θ_k we have $\vartheta(\Theta_k) = \Theta_{-k}$.

Next for $\lambda\in\mathbb{Z}\delta$ let $\omega : W(\lambda) \longrightarrow W(\lambda)$ be the unique linear map satisfying $\omega(e^{\lambda}\otimes x) := e^{\lambda}\otimes\vartheta(x)$, $x\in\Theta$. Thus ω is an involution of $W(\lambda)$ satisfying $\omega(W_k(\lambda)) = W_{-k}(\lambda)$. Finally define a new Hermitian form $\langle\cdot|\cdot\rangle$ on $W(\lambda)$ in the following way: For $x,x'\in\Theta$ and $z = e^{\lambda}\otimes x$, $z' = e^{\lambda}\otimes x' \in W(\lambda)$, set

$$\langle z|z'\rangle := (x|\omega(x'))$$

The *square length* of an element $z \in W(\lambda)$ is defined by $\|z\|^2 := \langle z|z\rangle$.

Proposition 2.5.3: (i) The set $\{z = e^{\lambda}\otimes\mu(-n)\delta(-m) : n\in Z_+^s,\ m\in Z_+^r,\ r,s\geq 0\}$ is an orthogonal basis for $W(\lambda)$ with respect to $\langle\cdot|\cdot\rangle$.

(ii) $\langle\cdot|\cdot\rangle$ is positive definite on $W(\lambda)$ and if $z = e^{\lambda}\otimes\mu(-n)\delta(-m)$ then $\|z\|^2 = c(n)c(m)\pi(n)\pi(m)$.

Proof: (i) It is clear that the stated set of elements is a basis for $W(\lambda)$. Let $z = e^{\lambda}\otimes\mu(-n)\delta(-m)$ and $z' = e^{\lambda}\otimes\mu(-n')\delta(-m')$. Then

$$\begin{aligned}\langle z|z'\rangle &= (\mu(-n)\delta(-m)|\vartheta(\mu(-n')\delta(-m')))\\ &= (\mu(-n)\delta(-m)|\mu(-m')\delta(-n'))\\ &= (\mu(-n)|\delta(-n'))\ (\mu(-m')|\delta(-m))\\ &= \delta_{n,n'}\ c(n)\pi(n)\ \delta_{m,m'}\ c(m)\pi(m).\end{aligned}$$

Suppose $z\neq z'$. Then either $(n)\neq(n')$ or $(m)\neq(m')$ and hence either $\delta_{n,n'}=0$ or $\delta_{m,m'}=0$. Now (i) is clear by the above calculation. Finally, if $z=z'$ then $\delta_{n,n'} = 1 = \delta_{m,m'}$ so (ii) follows. ∎

§2.6 Unitarity

Recall that a unitary module over $\mathcal{V}ir$ is one which admits a positive definite Hermitian form with respect to which the action of $\mathcal{V}ir$ is contravariant. Now the usual choice of such a form for Lie algebras with triangular decomposition is the Shapavalov form. However, as discussed in §1.5, the Shapavalov form $(\cdot|\cdot)$ on $W(\lambda)$ is not positive definite and hence, with respect to it, $W(\lambda)$ is not unitary. The goal of this section is to show that the $\mathcal{V}ir$-modules $W_k(\lambda)$, $k\in\mathbb{Z}$, $\lambda\in\mathbb{Z}\delta$ are unitary with respect to the (induced) form $\langle\cdot|\cdot\rangle$ constructed in §2.5. We begin by showing that the whole module $W(\lambda)$ is not unitary with respect to this form.

Example: For $\lambda\neq 0$ we have $\langle H_{-1}\cdot e^{\lambda}\otimes\mu(-1)\delta(-1)|e^{\lambda}\otimes\mu(-1)\delta(-1)^2\rangle = 1$ whereas $\langle e^{\lambda}\otimes\mu(-1)\delta(-1)|H_1\cdot e^{\lambda}\otimes\mu(-1)\delta(-1)^2\rangle = 0$. Thus the action of $\mathcal{V}ir$ on $W(\lambda)$ is not contravariant with respect to $\langle\cdot|\cdot\rangle$ and σ.

Proposition 2.6.1: The action of $\mathcal{V}ir$ on $\overline{W_k(0)}$, $k\in\mathbb{Z}$ is contravariant with respect to the form $\langle\cdot|\cdot\rangle$ and the involution σ.

Proof: Fix $k\in\mathbb{Z}$. Note that by the table in §2.4 and proposition 2.5.3 we only have to consider the following two cases. (I) $\mathbf{n},\mathbf{n}'\in\mathbb{Z}_+^S$ and $\mathbf{m},\mathbf{m}'\in\mathbb{Z}_+^r$ and (II) $\mathbf{n}\in\mathbb{Z}_+^S$, $\mathbf{n}'\in\mathbb{Z}_+^{S+1}$, $\mathbf{m}\in\mathbb{Z}_+^r$ and $\mathbf{m}'\in\mathbb{Z}_+^{r+1}$. To prove (I),

$\langle H_k\cdot e^0\otimes\mu(-\mathbf{n})\delta(-\mathbf{m})|e^0\otimes\mu(-\mathbf{n}')\delta(-\mathbf{m}')\rangle = \langle e^0\otimes\mu(-\mathbf{n})\delta(-\mathbf{m})|H_{-k}\cdot e^0\otimes\mu(-\mathbf{n}')\delta(-\mathbf{m}')\rangle$

We compute LS = $\langle H_k\cdot e^0\otimes\mu(-\mathbf{n})\delta(-\mathbf{m})|e^0\otimes\mu(-\mathbf{n}')\delta(-\mathbf{m}')\rangle$

$$= \sum_{p=1}^{s} n_p \, (\mu(-n_1)\cdots\mu(k-n_p)\cdots\mu(-n_s)\delta(-\mathbf{m})|\mu(-\mathbf{m}')\delta(-\mathbf{n}'))$$

$$+ \sum_{q=1}^{r} m_q \, (\mu(-\mathbf{n})\delta(-m_1)\cdots\delta(k-m_q)\cdots\delta(-m_r)|\mu(-\mathbf{m}')\delta(-\mathbf{n}'))$$

where it is understood that all summands are zero if $k \geq n_p$ or $k \geq m_q$ for $1 \leq p \leq s$ and $1 \leq q \leq r$. Note that all terms appearing in the definition of H_k but not appearing in this expression are zero since distinct monomials are orthogonal (proposition 2.5.3).

Next for $1 \leq p \leq s$ and $1 \leq q \leq r$ we introduce the elements

$\mathbf{n}_{p_\pm} := (n_1,\ldots,\mp k+n_p,\ldots,n_s)$ and

$\mathbf{m}_{q_\pm} := (m_1,\ldots,\mp k+m_q,\ldots,m_r)$.

Then $\mu(-\mathbf{n}_{p_\pm}) = \mu(-n_1)\cdots\mu(\pm k-n_p)\cdots\mu(-n_s)$

and $\delta(-\mathbf{m}_{q_\pm}) = \delta(-m_1)\cdots\delta(\pm k-m_q)\cdots\delta(-m_r)$.

With this notation we have (for $k \neq 0$)

$$LS = \begin{cases} \sum_{p=1}^{s} n_p(\mu(-\mathbf{n}_{p_+})\delta(-\mathbf{m})|\mu(-\mathbf{m}')\delta(-\mathbf{n}')), & \mathbf{m}=\mathbf{m}' \\ \sum_{q=1}^{r} m_q(\mu(-\mathbf{n})\delta(-\mathbf{m}_{q_+})|\mu(-\mathbf{m}')\delta(-\mathbf{n}')), & \mathbf{m}\neq\mathbf{m}'. \end{cases}$$

If $k=0$ then $\mathbf{n}_{p_+} = \mathbf{n}$ and $\mathbf{m}_{q_+} = \mathbf{m}$ so that in this case

$$LS = \sum_{p=1}^{s} n_p(\mu(-\mathbf{n})\delta(-\mathbf{m})|\mu(-\mathbf{m}')\delta(-\mathbf{n}')) + \sum_{q=1}^{r} m_q(\mu(-\mathbf{n})\delta(-\mathbf{m})|\mu(-\mathbf{m}')\delta(-\mathbf{n}')).$$

Now if $k\neq 0$ then $RS = \langle e^0\otimes\mu(-\mathbf{n})\delta(-\mathbf{m})|H_{-k}\cdot(e^0\otimes\mu(-\mathbf{n}')\delta(-\mathbf{m}')\rangle$

$$= \sum_{p=1}^{s} n_p'(e^0\otimes\mu(-\mathbf{n})\delta(-\mathbf{m})|e^0\otimes\mu(-\mathbf{m}')\delta(-n_1')\cdots\delta(-k-n_p')\cdots\delta(-n_s'))$$

$$+ \sum_{q=1}^{r} m_q' \ (e^0\otimes\mu(-\mathbf{n})\delta(-\mathbf{m})|e^0\otimes\mu(-m_1')\cdots\mu(-k-m_q')\cdots\mu(-m_r')\delta(-\mathbf{n}'))$$

$$= \begin{cases} \sum_{p=1}^{r} n_p' \ (\mu(-\mathbf{n})\delta(-\mathbf{m})|\mu(-\mathbf{m}')\delta(-n_{p_}')), & \mathbf{m}=\mathbf{m}' \\ \sum_{q=1}^{s} m_q' \ (\mu(-\mathbf{n})\delta(-\mathbf{m})|\mu(-m_{q_}')\delta(-\mathbf{n}')), & \mathbf{m}\neq\mathbf{m}' \end{cases}$$

If $k=0$ then $\mathbf{n}_{p_}' = \mathbf{n}'$ and $\mathbf{m}_{q_}' = \mathbf{m}'$ and so in this case

$$RS = \sum_{p=1}^{s} n_p'(\mu(-\mathbf{n})\delta(-\mathbf{m})|\mu(-\mathbf{m}')\delta(-\mathbf{n}')) + \sum_{q=1}^{r} m_q'(\mu(-\mathbf{n})\delta(-\mathbf{m})|\mu(-\mathbf{m}')\delta(-\mathbf{n}')).$$

Now when $k=0$ either (a) $\mathbf{m}\neq\mathbf{m}'$ or $\mathbf{n}\neq\mathbf{n}'$ in which case LS = 0 = RS by 2.5.1 (iii) or (b) $\mathbf{m}=\mathbf{m}'$ and $\mathbf{n}=\mathbf{n}'$ in which case LS = RS by inspection. Thus it suffices to show the following:

$$\text{(i)} \ \sum_{p=1}^{s} n_p(\mu(-\mathbf{n}_{p_+})|\delta(-\mathbf{n}')) = \sum_{p=1}^{s} n_p'(\mu(-\mathbf{n})|\delta(-\mathbf{n}_{p_}')) \quad \text{and}$$

$$\text{(ii)} \ \sum_{q=1}^{r} m_q(\delta(-\mathbf{m}_{q_+})|\mu(-\mathbf{m}')) = \sum_{q=1}^{r} m_q'(\delta(-\mathbf{m})|\mu(-\mathbf{m}_{q_}')).$$

We prove only (i) since (ii) is similar. Recall that $k\in\mathbb{Z}$, $\mathbf{n},\mathbf{n}'\in\mathbb{Z}_+^s$ and $\mathbf{n}_{p_\pm} = (n_1,\ldots,\mp k+n_p,\ldots,n_s)$. Now introduce the sets

$$I_+(\mathbf{n},\mathbf{n}') := \left\{ 1\le p\le s : \langle \mathbf{n}_{p_+} \rangle = \langle \mathbf{n}' \rangle \right\}$$

$$I_-(\mathbf{n},\mathbf{n}') := \left\{ 1\le p\le s : \langle \mathbf{n}'_{p_-} \rangle = \langle \mathbf{n} \rangle \right\} .$$

Note that as ordered s-tuples $\mathbf{n}_{p_+} = \mathbf{n}' \iff \mathbf{n}'_{p_-} = \mathbf{n}$ but this does not imply $I_+(\mathbf{n},\mathbf{n}') = I_-(\mathbf{n},\mathbf{n}')$. For example, consider $\mathbf{n}$: $= (n_1,n_2,n_3) = (2,4,8)$ and $\mathbf{n}' := (n'_1,n'_2,n'_3) = (2,2,8)$ with $k=2$. Then $I_+(\mathbf{n},\mathbf{n}') = \{2\}$ but $I_-(\mathbf{n},\mathbf{n}') = \{1,2\}$. Finally, we point out that $I_+(\mathbf{n},\mathbf{n}') = \varnothing \iff I_-(\mathbf{n},\mathbf{n}') = \varnothing$. Now, for (i), we have

$$LS = \sum_{p=1}^{s} n_p(\mu(-\mathbf{n}_{p_+})|\delta(-\mathbf{n}')) = \sum_{p=1}^{s} n_p\, \delta_{\mathbf{n}_{p_+},\mathbf{n}'}\, c(\mathbf{n}')\pi(\mathbf{n}')$$

$$= \sum_{p\in I_+(\mathbf{n},\mathbf{n}')} n_p\, c(\mathbf{n}')\pi(\mathbf{n}') \quad \text{and}$$

$$RS = \sum_{p=1}^{s} n_p'(\mu(-\mathbf{n})|\delta(-\mathbf{n}'_{p_-})) = \sum_{p=1}^{s} n'_p\, \delta_{\mathbf{n},\mathbf{n}'_{p_-}}\, c(\mathbf{n})\pi(\mathbf{n})$$

$$= \sum_{p\in I_-(\mathbf{n},\mathbf{n}')} n'_p\, c(\mathbf{n})\pi(\mathbf{n}).$$

Thus it suffices to show

$$\left(\sum_{p\in I_+(\mathbf{n},\mathbf{n}')} n_p \right) c(\mathbf{n}')\pi(\mathbf{n}') = \left(\sum_{p\in I_-(\mathbf{n},\mathbf{n}')} n'_p \right) c(\mathbf{n})\pi(\mathbf{n}).$$

We will assume that $I_{\pm}(\mathbf{n},\mathbf{n}') \neq \varnothing$ otherwise both sides are zero. Now write

$$\mathbf{n} = (\underbrace{n_1,\ldots n_1}_{a_1\text{-times}},\ldots,\underbrace{n_p,\ldots,n_p}_{a_p\text{-times}},\ldots,\underbrace{n_v,\ldots,n_v}_{a_v\text{-times}})$$

where $n_i<n_j$ if $i<j$. Note $I_{\pm}(\mathbf{n},\mathbf{n}') \neq \varnothing$ implies that $\mathbf{n}$ coincides with $\mathbf{n}'$ except in exactly one entry. That is, if we write

$$\mathbf{n}' = (\underbrace{n'_1,\ldots,n'_1}_{a'_1\text{-times}},\ldots,\underbrace{n'_q,\ldots,n'_q}_{a'_q\text{-times}},\ldots\underbrace{n'_w,\ldots,n'_w}_{a'_w\text{-times}}),$$

then there is exactly one more n_p in $\mathbf{n}$ than in $\mathbf{n}'$ (for some $1\le p\le v$) and exactly one more n'_q in $\mathbf{n}'$ than in $\mathbf{n}$ (for some $1\le q\le w$). Thus we have

$$\frac{\pi(\mathbf{n})}{\pi(\mathbf{n}')} = \frac{n_p}{n'_q} \quad \text{and} \quad \frac{c(\mathbf{n})}{c(\mathbf{n}')} = \frac{a_p}{a'_q}.$$

Finally, $$\frac{\sum_{p\in I_+(\mathbf{n},\mathbf{n}')} n_p}{\sum_{p\in I_-(\mathbf{n},\mathbf{n}')} n'_p} = \frac{\text{card } I_+(\mathbf{n},\mathbf{n}')\, n_p}{\text{card } I_-(\mathbf{n},\mathbf{n}')\, n'_q} = \frac{a_p n_p}{a'_q n'_q} = \frac{c(\mathbf{n})\pi(\mathbf{n})}{c(\mathbf{n}')\pi(\mathbf{n}')}.$$

Next we consider the case (II) where $\mathbf{n} \in \mathbb{Z}_+^s$, $\mathbf{n}' \in \mathbb{Z}_+^{s+1}$, $\mathbf{m} \in \mathbb{Z}_+^r$ and $\mathbf{m}' \in \mathbb{Z}_+^{r+1}$. We need to show that for $k>0$,

$$\langle H_k(e^0\otimes\mu(-\mathbf{n}')\delta(-\mathbf{m}'))\,|\,(e^0\otimes\mu(-\mathbf{n})\delta(-\mathbf{m}))\rangle = \langle(e^0\otimes\mu(-\mathbf{n}')\delta(-\mathbf{m}'))\,|\,H_{-k}(e^0\otimes\mu(-\mathbf{n})\delta(-\mathbf{m}))\rangle$$

Both sides of this equation are zero unless for $p,q>0$ with $p+q=k$ we have $\mathbf{n}' = (n_1,\ldots,p,\ldots,n_s)$ and $\mathbf{m}' = (m_1,\ldots,q,\ldots,m_r)$ where $\mathbf{n} = (n_1,\ldots,n_s)$ and $\mathbf{m} = (m_1,\ldots,m_r)$. Thus $\mathbf{n}$ and $\mathbf{n}'$ differ only in one component and similarly for $\mathbf{m}$ and $\mathbf{m}'$. Now suppose

$$\mathbf{n} = (\underbrace{n_1,\ldots,n_1}_{a_1\text{-times}},\ldots,\underbrace{p,\ldots,p}_{a\text{-times}},\ldots,\underbrace{n_v,\ldots,n_v}_{a_v\text{-times}}) \text{ and}$$

$$\mathbf{m} = (\underbrace{m_1,\ldots,m_1}_{b_1\text{-times}},\ldots,\underbrace{q,\ldots,q}_{b\text{-times}},\ldots,\underbrace{m_w,\ldots,m_w}_{b_w\text{-times}}) \text{ where } b_i,a_j,a,b\geq 0 \text{ and } 1\leq v\leq s,\ 1\leq w\leq r.$$

Then $\dfrac{c(\mathbf{n}')}{c(\mathbf{n})} = a+1 = (\#\ p\text{'s in } \mathbf{n}')$ and

$\dfrac{c(\mathbf{m}')}{c(\mathbf{m})} = b+1 = (\#\ q\text{'s in } \mathbf{m}')$ and hence for $\varepsilon \in \{1,2\}$ we have

$$\begin{aligned}
LS &= \langle H_k(e^0\otimes\mu(-\mathbf{n}')\delta(-\mathbf{m}')\,|\,e^0\otimes\mu(-\mathbf{n})\delta(-\mathbf{m})\rangle \\
&= \varepsilon/2\ (\#\ p\text{'s in } \mathbf{n}')p\cdot(\#\ q\text{'s in } \mathbf{m}')q\cdot\|e^0\otimes\mu(-\mathbf{n})\delta(-\mathbf{m})\|^2 \\
&= \varepsilon/2\ (a+1)p\cdot(b+1)q\cdot c(\mathbf{m})\pi(\mathbf{m})\cdot c(\mathbf{n})\pi(\mathbf{n}) \text{ and} \\
RS &= \langle e^0\otimes\mu(-\mathbf{n}')\delta(-\mathbf{m}')\,|\,H_{-k}(e^0\otimes\mu(-\mathbf{n})\delta(-\mathbf{m})\rangle \\
&= \varepsilon/2\ \|e^0\otimes\mu(-\mathbf{n}')\delta(-\mathbf{m}')\|^2 \\
&= \varepsilon/2\ c(\mathbf{n}')c(\mathbf{m}')\pi(\mathbf{n}')\pi(\mathbf{m}') \\
&= \varepsilon/2\ c(\mathbf{n})(a+1)\cdot c(\mathbf{m})(b+1)\cdot\pi(\mathbf{n})p\cdot\pi(\mathbf{m})q = LS. \qquad \blacksquare
\end{aligned}$$

Corollary 2.6.2: For $k\in\mathbb{Z}$ and $\lambda\in\mathbb{Z}\delta$ the *Vir*-module $W_k(\lambda)$ is unitary.

Proof: This follows immediately by first transfering the form from $\overline{W_k(0)}$ to $W_k(\lambda)$ and then applying propositions 2.4.2 and 2.6.1 . $\blacksquare$

§2.7 Complete Reducibility

We are finally in a position to prove the main result of the chapter.

Proposition 2.7.1: For every $k\in\mathbb{Z}$ the $\mathcal{V}ir$-modules $\overline{\overline{W_k(0)}}$ are completely reducible.

Proof: For convenience we will denote the module $\overline{\overline{W_k(0)}}$ by $\overline{W}_k$. Fix any $k\in\mathbb{Z}$ and let $\overline{W}_k(j)$ denote the subspace of the submodule $\overline{W}_k$ spanned by all elements of degree j. For $\mathbf{n} := (n_1,\dots,n_s) \in \mathbb{Z}_+^s$ and $\mathbf{m} := (m_1,\dots,m_r) \in \mathbb{Z}_+^r$ the element $x := e^0\otimes\mu(-\mathbf{n})\delta(-\mathbf{m}) \in \overline{W}_k$ has degree $\sum_{i=1}^{r} m_i + \sum_{i=1}^{s} n_i$. It is clear then that $\overline{W}_k = \coprod_{j\geq 0} \overline{W}_k(j)$.

Since the operator $\delta(0)$ kills the module $\overline{W}_k$, the operator H_0 can be written $H_0 = \sum_{j>0} \big(\mu(-j)\delta(j) + \delta(-j)\mu(j)\big)$ and we claim that the eigenspace of H_0 with eigenvalue j is exactly $\overline{W}_k(j)$. That is, $\overline{W}_k(j) = \{x \in \overline{W}_k : H_0(x) = jx\}$. It suffices to show that if $x\in\overline{W}_k$ then $H_0x = (\deg x)x$. Indeed, for $x := \mu(-\mathbf{n})\delta(-\mathbf{m})$ we have

$$\begin{aligned} H_0\cdot x &= \sum_{\ell\geq 0} \big(\mu(-\ell)\delta(\ell) + \delta(-\ell)\mu(\ell)\big)\cdot x \\ &= \Big(\sum_{i=1}^{s} n_i + \sum_{i=1}^{r} m_i\Big)x \\ &= (\deg x)x. \end{aligned}$$

Note that the decomposition $\overline{W}_k = \coprod_{j\geq 0} \overline{W}_k(j)$ is actually a weight space decomposition of $\overline{W}_k$ with respect to the commutative subalgebra $\mathbb{C}d_0\oplus\mathbb{C}\mathfrak{z}$. It is a standard result that any $\mathcal{V}ir$-module admitting a weight space decomposition

with respect to $\mathbb{C}d_0 \oplus \mathbb{C}$; has the property that any submodule respects this decomposition. For a detailed proof of this see [KR] lemma 1.1. In our case, if $\overline{U}_k$ is a $\mathcal{V}ir$-submodule of $\overline{W}_k$ then we have $\overline{U}_k = \coprod_{j\geq 0} \overline{U}_k(j)$ where $\overline{U}_k(j) := \overline{W}_k(j) \cap \overline{U}_k$. Since $\overline{W}_k(j)$ is finite dimensional (we will establish a formula for its dimension in the next section) and since the Hermitian form $\langle \cdot | \cdot \rangle$ is positive definite on $\overline{W}_k(j)$, there is a subspace $\overline{U}_k(j)^{\perp}$ of $\overline{W}_k(j)$ satisfying $\overline{W}_k(j) = \overline{U}_k(j) \oplus \overline{U}_k(j)^{\perp}$ and $\langle \overline{U}_k(j) | \overline{U}_k(j)^{\perp} \rangle = 0$.

Define $\overline{U}_k^{\perp} := \coprod_{j\geq 0} \overline{U}_k(j)^{\perp}$. We claim that $\overline{U}_k^{\perp} = \{x \in \overline{W}_k : \langle \overline{U}_k | x \rangle = 0\}$. For this we need to observe that $\langle \overline{W}_k(i) | \overline{W}_k(j) \rangle = 0$,$\forall i \neq j$. Indeed, if $z = e^0 \otimes \mu(-n)\delta(-m) \in \overline{W}_k(i)$ and $z' = e^0 \otimes \mu(-n')\delta(-m') \in \overline{W}_k(j)$ then $i \neq j$ implies $(n) \neq (n')$ or $(m) \neq (m')$ in which case $\langle z | z' \rangle = 0$ by propositon 2.5.3.. To establish the claim let $x \in \overline{U}_k^{\perp} = \coprod_{j\geq 0} \overline{U}_k(j)^{\perp}$ and write $x = \sum_i x_i$, $x_i \in \overline{U}_k^{\perp}(i)$. Similarly let $y \in \overline{U}_k$ and write $y = \sum_j y_j$, $y_j \in \overline{U}_k(j)$. Then $\langle x | y \rangle = \sum_{i,j} \langle x_i | y_j \rangle$. Now for $i=j$, $\langle x_i | y_i \rangle = 0$ since $\langle \overline{U}_k(i) |, \overline{U}_k(i)^{\perp} \rangle = 0$. For $i \neq j$, $\langle x_i | y_j \rangle \in \langle \overline{W}_k(i) | \overline{W}_k(j) \rangle = 0$. Thus $\langle x | y \rangle = 0$ and hence $\langle x | \overline{U}_k \rangle = 0$. For the reverse inclusion, let $x \in \overline{W}_k$ satisfy $\langle x | \overline{U}_k \rangle = 0$. Write $x = \sum_i x_i$, $x_i \in \overline{W}_k(i)$. We need to show that $x_i \in \overline{U}_k(i)^{\perp}$. Let $y \in \overline{U}_k(i) = \overline{U}_k \cap \overline{W}_k(i)$. Then $\langle x_i | y \rangle = \langle x - \sum_{j\neq i} x_j | y \rangle = \langle x | y \rangle - \sum_{j\neq i} \langle x_j | y \rangle = 0$ since $\langle \overline{W}_k(i) | \overline{W}_k(j) \rangle = 0$, $i \neq j$. This proves the claim.

Next since $\overline{U}_k$ is a $\mathcal{V}ir$-submodule, $H_n \cdot \overline{U}_k \subset \overline{U}_k$ for any $n \in \mathbb{Z}$. But then by proposition 2.6.1, $0 = \langle H_n \cdot \overline{U}_k | \overline{U}_k^{\perp} \rangle = \langle \overline{U}_k | H_{-n} \cdot \overline{U}_k^{\perp} \rangle$ and hence $\overline{U}_k^{\perp}$ is also a $\mathcal{V}ir$-submodule of $\overline{W}_k$.

In summary we have the submodule decomposition $\overline{W}_k = \overline{U}_k \oplus \overline{U}_k^{\perp}$ and hence the lattice of submodules is complemented. This means that $\overline{W}_k$ is completely reducible. ∎

Corollary 2.7.2: W(0) is completely reducible.

Proof: This follows from propositions 2.4.1 (ii) and 2.7.1. ∎

Corollary 2.7.3: For any $k \in \mathbb{Z}$ and $\lambda \in \mathbb{Z}\delta$ the modules $W_k(\lambda)$ are completely reducible.

Proof: This follows immediately from propositions 2.4.2 and 2.7.1. ∎

§2.8 The Structure of the Irreducible Components

Unless otherwise stated, in this section we fix $k\in\mathbb{Z}$ and $\lambda\in\mathbb{Z}\delta$ and denote $W_k(\lambda)$ simply by W_k. We first reduce the question of determining the structure of the irreducible components of any W_k to that of determining the set of highest weight vectors, which we call vacuum vectors. We find a character formula for the module W_k which, together with the known character formulas for the Verma modules $M(c,h)$ of $\mathcal{V}ir$, allows us to give a procedure for developing a decomposition of W_k into irreducible highest weight modules. We illustrate the method when $k=-1$ and we indicate how to find the explicit vacuum vectors. Determining the vacuum vectors in the way we do is impractical but will always work. The case $k=0$ is slightly different but leads to an application of the character formula for W_k. Namely, we show that the Verma module $M(2,0)$ is not irreducible.

Proposition 2.8.1: Fix a submodule U_k of the unitary $\mathcal{V}ir$-module W_k. Then U_k is a highest weight module if and only if it is irreducible.

Proof: Suppose U_k is not irreducible. Let N_k be a proper nonzero submodule of U_k. Since W_k is unitary, so is U_k and hence by the arguments in the previous section we have a submodule decomposition $U_k = N_k\oplus N_k^{\perp}$. Now if U_k is a highest weight module over $\mathcal{V}ir$ then its highest weight space is one-dimensional spanned by a vacuum vector, say v_k, and v_k generates U_k. Moreover, $d_0\cdot v_+ = hv_+$ and $\mathfrak{z}\cdot v_+ = cv_+$ for some $c,h\in\mathbb{C}$. Since N_k is a proper submodule of U_k it does not contain v_k. Moreover, since v_+ spans the h-eigenspace of d_0 it belongs to exactly one summand and hence $v_k \in N_k^{\perp}$. This

implies $N_k^{\perp} = U_k$, a contradiction.

($\Leftarrow$) Conversely suppose U_k is irreducible. Consider the weight space decomposition $W_k = \coprod_{j\geq 0} W_k(j)$. Then U_k respects this decomposition and can be written $U_k = \coprod_{j\geq 0} U_k(j)$ where $U_k(j) := U_k \cap W_k(j)$. Let j_0 be the smallest integer j such that $U_k(j) \neq 0$. Let $0 \neq v_k \in U_k(j_0)$. Then v_k is a vacuum vector in U_k and it generates all of U_k since U_k is irreducible. ∎

Corollary 2.8.2: (i) For $k\neq 0$ $W_k \cong \coprod_{h>0} r_h(k)\, L(2,h)$ and $L(2,h) = M(2,h)$ $\forall h>0$ where $r_h(k)$ denotes the multiplicity of $L(2,h)$ in this decomposition.

(ii) $W_0 = L(2,0) \oplus \left(\coprod_{h>0} r_h(0)\, L(2,h) \right)$.

Proof: This follows immediately from propositions 2.1.1, 2.7.1 and 2.8.1. ∎

Our next goal is to find $\dim_{\mathbb{C}} W_k(n)$ where $W_k(n) = \{x\in W_k : L_0 x = nx\}$. Let $p_r(s)$ denote the number of partitions of s into r parts. It is a well-known fact (see Hardy [H] - equation (19.3.7)) that $p_r(s)$ is the coefficient of q^s in the expansion of $\dfrac{q^r}{(1-q)(1-q^2)\cdots(1-q^r)}$.

Proposition 2.8.3: For $k\in\mathbb{Z}$ and $n\geq 0$ we have

$$\dim_{\mathbb{C}} W_k(n) = \sum_{j\geq 0} \left(\sum_{0\leq m\leq n} p_{j+k}(m)\, p_j(n-m) \right) < \infty. \tag{2.8.1}$$

Proof: For any $k>0$ note that there is a one-to-one correspondence between the partitions of k into r parts, i.e. elements $\mathbf{m} = (m_1,\ldots,m_r) \in \mathbb{Z}_+^r$ with $\sum_{i=1}^{r} m_i = k$, and monomials $\delta(-\mathbf{m}) := \delta(-m_1)\cdots\delta(-m_r) \in \Delta_r$ of degree k.

Recall that, as a $\mathbb{C}$-space, $W_k \cong \mathbb{C}e^{\lambda} \otimes_{\mathbb{C}} \left(\coprod_{j\geq 0} \Omega_{j+k}\Delta_j \right)$. Now equation (2.8.1) counts the number of partitions of n of the following type. For every $j \geq 0$ and $0 < m \leq n$ we are interested in partitions of n into (2j+k) parts by first partitioning m into (j+k) parts in one color, say blue (corresponding to Ω_{j+k}) and then partitioning the remaining n-m into j parts in another color, say red (corresponding to Δ_j). Of course, for every $n>0$ there are only finitely many j satisfying these conditions. Thus $\dim_{\mathbb{C}} W_k(n) < \infty$, $\forall n,k$. ∎

Example: We will discuss the structure of W_{-1}. First we illustrate formula (2.8.1) by computing $\dim_{\mathbb{C}} W_{-1}(5)$.

Since $W_{-1} \cong \mathbb{C}e^{\lambda} \otimes_{\mathbb{C}} \left((\Omega_0\Delta_1) \oplus (\Omega_1\Delta_2) \oplus (\Omega_2\Delta_3) \oplus \cdots \right)$, clearly only j = 1,2 and 3 in formula (2.8.1) contribute to the sum and we have

$$\dim_{\mathbb{C}} W_{-1}(5) = \sum_{j=1}^{3} \left(\sum_{m=0}^{5} p_{j-1}(m)p_j(5-m) \right)$$

$$= \left(\sum_{m=0}^{5} p_0(m)p_1(5-m) \right) + \left(\sum_{m=0}^{5} p_1(m)p_2(5-m) \right) + \left(\sum_{m=0}^{5} p_2(m)p_3(5-m) \right)$$

$$= (1) + (2+1+1) + (1) = 6.$$

Let us explicitly write out the six basis vectors. For $\mathbf{n} \in \mathbb{Z}_+^s$ and $\mathbf{m} \in \mathbb{Z}_+^r$ we denote $e^{\lambda} \otimes \mu(-\mathbf{n})\delta(-\mathbf{m})$ by $\mathbf{n},\mathbf{m}$ so, for example, the vector $e^{\lambda} \otimes \mu(-2)\delta(-2)\delta(-1)$ is denoted by 2,21 or by (2,21). Thus a basis for $W_{-1}(5)$ is given by:

$$\begin{cases} x_1 := (0,5) \ , & x_4 := (1,31) \\ x_2 := (3,11) & x_5 := (1,22) \\ x_3 := (2,21) & x_6 := (11,111) \end{cases}$$

In a similar way, one shows the following using (2.8.1):

space	$W_{-1}(0)$	$W_{-1}(1)$	$W_{-1}(2)$	$W_{-1}(3)$	$W_{-1}(4)$	$W_{-1}(5)$	$W_{-1}(6)$
$\dim_{\mathbb{C}}$	0	1	1	2	3	6	9

Next we use this information together with the character formula

ch $M(c,h) = \sum_{j>0} p(j)q^{h+j}$ to illustrate the procedure for determining the decomposition of W_{-1} into irreducible Verma modules. From the table above we know that ch $W_{-1} = q + q^2 + 2q^3 + 3q^4 + 6q^5 + 9q^6 + \cdots$. One easily checks that $e^{\lambda}\otimes\delta(-1) \in W_{-1}$ is a vacuum vector and hence the irreducible Verma module that it generates has character

$$\text{ch } M(2,1) = \sum_{j\geq 0} p(j)q^{j+1} = q + q^2 + 2q^3 + 3q^4 + 5q^5 + 7q^6 + \cdots .$$

By comparison of the two characters, we deduce that there must be a vacuum vector in $W_{-1}(5)$ and $W_{-1} = M(2,1) \oplus M(2,5) \oplus \cdots$. Since ch $M(2,5) = q^5 + q^6 + 2q^7 + \cdots$, we see that there is a single vacuum vector in $W_{-1}(6)$ and $W_{-1} = M(2,1) \oplus M(2,5) \oplus M(2,6) \oplus \cdots$. One can, of course, continue in this way.

What remains untold here is the explicit form of the vacuum vectors in $W_{-1}(5)$, $W_{-1}(6)$, etc.. We describe a tedious method for finding them and we compute the first few. This method always works in principle but quickly becomes very difficult to implement by hand. First note that by the commutation relations defining the Virasoro algebra, a vector x in a highest weight module for $\mathcal{V}ir$ is a vacuum vector if and only if $d_1 \cdot x = 0 = d_2 \cdot x$.

Now consider $W_{-1}(5)$ and let $x = \sum_{i=1}^{6} a_i x_i$ be an arbitrary element in this space. Writing out the equations $d_1 \cdot x = 0 = d_2 \cdot x$ we find that $a_1 = 0 = a_4$ and $\frac{3}{2} a_2 + a_3 = 0$, $a_3 + 2a_5 = 0$ and $\frac{3}{2} a_2 + a_6 = 0$. Solving these produces the vacuum vector $\overline{e^{\lambda} \otimes (4x_1 - 6x_3 + 3x_5 - 6x_6)}$

Finally we consider $W_{-1}(6)$ with basis:

$$\begin{cases} y_1 := (0,6) & y_5 := (2,22) \\ y_2 := (4,11) & y_6 := (1,41) \\ y_3 := (3,21) & y_7 := (1,32) \\ y_4 := (2,31) & y_8 := (21,111) \\ & y_9 := (11,211) \end{cases}$$

Now $y = \sum_{i=1}^{9} b_i y_i$ satisfying $d_1 \cdot y = 0 = d_2 \cdot y$ leads to $b_1 = 0 = b_6$ and the vacuum vector $\overline{e^{\lambda} \otimes (3y_2 - 6y_3 + 2y_4 + 3y_5 - 2y_7 - 9y_8 + 12y_9)}$.

We remark that the procedure above for determining the decomposition of W_{-1} works the same for all W_k provided $k\neq 0$. For $k>0$ (resp. $k<0$) the top weight space in the weight space decomposition of W_k is one dimensional spanned by the vacuum vector $e^{\lambda}\otimes\mu(-1)$ (resp. $e^{\lambda}\otimes\delta(-1)$). Since we know the character formula for the irreducible Verma module M(2,1) we can compare it to the character of W_k and determine where the next vacuum vector will appear, etc..

When k=0 this proceedure will not work since, as we will now show, $M(2,0) \neq L(2,0)$. Equation (2.8.1) gives ch $W_0 = 1 + q^2 + 2q^3 + 4q^4 + 6q^5 + \cdots$. Now the vacuum vector $e^{\lambda}\otimes 1$ spans the top weight space of W_0 and it generates the irreducible L(2,0) which cannot be a Verma module since if it were its character would be $1 + q + 2q^2 + \cdots$ which is impossible by comparison with ch W_0 above. However, one can still search for vacuum vectors in W_0 using the method that we used in the case of W_{-1}. We find that there is one vacuum vector at degree 3, namely $e^{\lambda}\otimes(1,2 - 2,1)$. At degree 4 there is no vacuum vector. However, at degree 5 we find the vacuum vector

$$e^{\lambda}\otimes\big((1,4 - 4,1) - 2(2,3 - 3,2) + 10(12,11 - 11,12)\big).$$

This allows us to approximate the character formula of L(2,0) up to degree 5:

$$\begin{aligned}\text{ch } L(2,0) &= \text{ch } W_0 - \text{ch } M(2,3) - \text{ch } M(2,5) - \cdots \\ &= (1 + q^2 + 2q^3 + 4q^4 + 6q^5 + \cdots) - (q^3 + q^4 + 2q^5 + \cdots) - (q^5 + \cdots) \\ &= 1 + q^2 + q^3 + 3q^4\ 3q^5 + \cdots .\end{aligned}$$

Proposition 2.8.4: For any $k\in\mathbb{Z}$ and any $\lambda\in\mathbb{Z}\delta$ $W_k(\lambda) \cong W_{-k}(\lambda)$ as $\mathcal{V}ir$-modules.

Proof: By proposition 2.4.2, if suffices to check that for any $k\in\mathbb{Z}$, $\overline{W_k(0)} \cong \overline{W_{-k}(0)}$. Indeed, we will show that the unique linear map (see §2.5) $\omega : \overline{W_k(0)} \longrightarrow \overline{W_{-k}(0)}$ satisfying $\omega(e^0\otimes\mu(-n)\delta(-m)) = e^0\otimes\mu(-m)\delta(-n)$ is an explicit isomorphism. Note that the map ω is the involution discussed in §2.5. It is clear that ω is a linear isomorphism. We need only check that $\omega(H_k\cdot e^0\otimes\mu(-n)\delta(-m)) = H_k\cdot e^0\otimes\mu(-m)\delta(-n)$. Indeed,

$$\omega\Big(H_k\cdot e^0\otimes\mu(-n)\delta(-m)\Big) = \omega\Big(\sum_{p=1}^{s} e^0\otimes\mu(-n_{p_+})\delta(-m) + \sum_{q=1}^{r} e^0\otimes\mu(-n)\delta(-m_{q_+})\Big)$$

$$= \sum_{p=1}^{s} e^0\otimes\mu(-m)\delta(-n_{p_+}) + \sum_{q=1}^{r} e^0\otimes\mu(-m_{q_+})\delta(-n)$$

$$= H_k\cdot e^0\otimes\mu(-m)\delta(-n). \qquad \blacksquare$$

REPRESENTATIONS OF THE SEMI-DIRECT PRODUCT LIE ALGEBRA $\tilde{\mathfrak{a}} := \mathcal{V}ir \propto \mathfrak{a}(Q)$

Introduction: In chapter 1 we introduced the generalized Heisenberg algebra $\mathcal{H}:=\mathcal{H}(\mathring{Q},2)$, its subalgebra $\mathfrak{L} \cong \mathfrak{a}(Q)$ and the corresponding canonical representation of $\mathfrak{a}(Q)$ on the Fock space $V_\Gamma(\lambda)$, $\lambda\in\Gamma\supset Q$.

In chapter 2 we studied the structure of some oscillator representations of $\mathcal{V}ir$ on the Fock space $V_\Lambda(\lambda)$, $\lambda\in p$.

In this chapter, we consider a representation of $\mathcal{V}ir$ on $\mathfrak{a}(Q)$ by derivations and we study the representation theory of the resulting semi-direct product algebra $\tilde{\mathfrak{a}} := \mathcal{V}ir \propto \mathfrak{a}(Q)$ on the Fock space $V_\Gamma(\lambda)$. We find that $V_\Gamma(\lambda)$ is an irreducible highest weight module when $\lambda\in\Gamma\setminus Q$ and when $\lambda\in Q$ it admits a filtration of submodules $\{V_k\}_{k\in\mathbb{Z}}$ arising from those studied in chapter 2) with completely reducible subquotients.

§ 3.1 Construction of the semi-direct product algebra

$$\tilde{\mathfrak{a}} := \mathcal{V}ir \propto \mathfrak{a}(Q)$$

Let L be a geometric lattice and $\mathfrak{a}(L)$ the corresponding Heisenberg algebra. Then $\mathfrak{a}(L)$ has linear basis $\{a_i(n);\ c : a_i \in L,\ n \in \mathbb{Z}\}$ and bracket relations as in (1.1.4).

We define a representation of $\mathcal{V}ir$ on $\mathfrak{a}(L)$ as follows. For every $k \in \mathbb{Z}$, let d_k be the unique derivation of $\mathfrak{a}(L)$ satisfying

$$\text{(3.1.2)} \qquad \begin{cases} d_k(a(n)) := -na(n+k) \\ d_k(c) := 0 \end{cases}$$

We also define $\mathfrak{z}(\mathfrak{a}(L)) \equiv 0$. One easily checks that $(\mathfrak{z}\, d_k - d_k\, \mathfrak{z})(a(n)) = 0 = [d_k, \mathfrak{z}](a(n))$ and $(d_k d_\ell - d_\ell d_k)(a(n)) = [d_k, d_\ell](a(n))$. This means that $\mathfrak{a}(L)$ is a module over $\mathcal{V}ir$. Note that, to this point, we have not used the fact that each d_k is a derivation of $\mathfrak{a}(L)$. Before moving on, we identify this representation as a special case of a class of well-known representations of $\mathcal{V}ir$.

Fix $\alpha, \beta \in \mathbb{C}$ and let $V := \coprod_{n \in \mathbb{Z}} \mathbb{C}v_n$ be an infinite dimensional $\mathbb{C}$-space with basis $\{v_i\}_{i \in \mathbb{Z}}$. Then for all $n, k \in \mathbb{Z}$ define

$$\text{(3.1.3)} \qquad \begin{cases} d_k(v_n) := -(n+\alpha+\beta+\beta k)v_{n+k} \\ \mathfrak{z}(v_n) := 0 \end{cases}$$

Then one checks that this defines a two-parameter family of representations of $\mathcal{V}ir$ with $\mathfrak{z}$ acting trivially. We denote this family by $V_{\alpha,\beta}$. Let us consider

the special case $\alpha = 0 = \beta$. It is easy to see that the only nontrivial submodule of $V_{0,0}$ is $\mathbb{C}v_0$ and so $V_{0,0}/\mathbb{C}v_0$ is an irreducible module over $\mathcal{V}ir$. It is also possible to show that $V_{0,0}$ is unitary but it is clearly not a highest weight module. From (3.1.3) it is clear that d_0 is diagonalizable and has eigenspaces of dimension at most one. Since we do not rely on any of these facts, we simply refer to [KR] for their proofs. In 1982, Kac published a conjecture that any irreducible representation of $\mathcal{V}ir$ on which d_0 is diagonalizable with finite dimensional weight spaces is either a highest weight module L(c,h), a lowest weight module $L(c,h)^*$, or a quotient of some $V_{\alpha,\beta}$. This has recently been confirmed independently by Mathieu and Martin-Piard.

Now by (3.1.2) it is clear that our $\mathcal{V}ir$-module $\mathfrak{a}(L)$ is a direct sum of $r = \text{rank } L$ copies of $V_{0,0}$.

Our main objective in this section is to construct a new Lie algebra from this representation called the semi-direct product algebra. As a $\mathbb{C}$-space, we let $\tilde{\mathfrak{a}}(L) := \mathcal{V}ir \oplus \mathfrak{a}(L)$. We make $\tilde{\mathfrak{a}}(L)$ into a Lie algebra in such a way that both $\mathcal{V}ir$ and $\mathfrak{a}(L)$ are subalgebras and the commutator of an element in $\mathcal{V}ir$ and an element in $\mathfrak{a}(L)$ is given by the representation. That is,

$$[d_k, a(n)] := d_k(a(n)) = -na(n+k).$$

It is in checking the Jacobi identity that one needs to use the fact that the d_k are derivations of $\mathfrak{a}(L)$. We denote the resulting algebra by $\tilde{\mathfrak{a}}(L) := \mathcal{V}ir \propto \mathfrak{a}(L)$. The purpose of this chapter is to develop a representation theory for $\tilde{\mathfrak{a}}(L)$ when $L = Q$. In this case, we denote the algebra $\tilde{\mathfrak{a}}(Q)$ simply by $\tilde{\mathfrak{a}}$.

From proposition 1.1.1 (iii) we know that the centre of $\mathfrak{a}(Q)$ is given by the $\mathbb{C}$-space $\mathfrak{h}(0) \oplus \mathbb{C}c \oplus \left(\coprod_{m\in\mathbb{Z}\setminus\{0\}} \mathbb{C}\delta(m) \right)$ where $\mathfrak{h} = \mathbb{C}\otimes_{\mathbb{Z}} Q$. Thus $\dim_{\mathbb{C}}\left(\text{centre } \mathfrak{a}(Q)\right) = \infty$. However, since $[d_k, \delta(m)] = -m\delta(m+k)$, the centre of $\tilde{\mathfrak{a}}$ is finite dimensional and it is easy to check

Proposition 3.1.1: centre $\tilde{\mathfrak{a}} = \mathfrak{h}(0) \oplus \mathbb{C}c \oplus \mathbb{C}\mathfrak{z}$ and hence $\dim_{\mathbb{C}}$centre $\tilde{\mathfrak{a}} = \ell+2$ where ℓ = rank $\dot{Q}$.

Next we show that $\tilde{\mathfrak{a}}$ admits a triangular decomposition in the full sense of [MP-2] and hence it will be meaningful to speak of Verma modules over $\tilde{\mathfrak{a}}$. Define $\tilde{\mathfrak{a}}_+$ (resp. $\tilde{\mathfrak{a}}_-$) to be the subalgebra of $\tilde{\mathfrak{a}}$ generated by $\{d_n, a(n) : a\in\mathfrak{h}, n>0\}$ (resp. $\{d_n, a(n) : a\in\mathfrak{h}, n<0\}$). Also define $\tilde{\mathfrak{a}}_0 := \mathbb{C}d_0\oplus\mathfrak{h}(0)\oplus\mathbb{C}c\oplus\mathbb{C}\mathfrak{z}$. Then $\tilde{\mathfrak{a}}_0$ is an abelian subalgebra. Next define $\tilde{\sigma} : \tilde{\mathfrak{a}} \longrightarrow \tilde{\mathfrak{a}}$ to be the unique anti-linear map satisfying $\tilde{\sigma}(d_n) := d_{-n}$, $\tilde{\sigma}(a(n)) := a(-n)$, $\tilde{\sigma}(\mathfrak{z}) := \mathfrak{z}$ and $\tilde{\sigma}(c) := c$ where $n\in\mathbb{Z}$ and $a\in\mathfrak{h}$. Then $\tilde{\sigma}$ is an anti-involution satisfying $\tilde{\sigma}(\tilde{\mathfrak{a}}_\pm) = \tilde{\mathfrak{a}}_\mp$ and $\tilde{\sigma}(\tilde{\mathfrak{a}}_0) = \tilde{\mathfrak{a}}_0$. Finally, we check that $\tilde{\mathfrak{a}}_+$ (and hence $\tilde{\mathfrak{a}}_-$ using $\tilde{\sigma}$) has a weight space decomposition with respect to $\tilde{\mathfrak{a}}_0$ under the adjoint representation. Define $\phi : \tilde{\mathfrak{a}}_0 \longrightarrow \mathbb{C}$ to be the unique $\mathbb{C}$-linear map satisfying $\phi(a(0)) := 0$, $\phi(d_0) := -1$ and $\phi(\mathfrak{z}) := 0 =: \phi(c)$. Now for $n>0$ define $\phi_n := n\phi \in \tilde{\mathfrak{a}}_0^*$ and $\tilde{\mathfrak{a}}_+^{\phi_n} := \{ x\in\tilde{\mathfrak{a}}_+ : [h,x] = \phi_n(h)x,\ h\in\tilde{\mathfrak{a}}_0 \}$. Then using the commutation relations in $\mathfrak{a}(Q)$ and in *Vir* one checks that $\tilde{\mathfrak{a}}_+^{\phi_n} = \mathbb{C}d_n \oplus \mathfrak{h}(n)$ and hence $\tilde{\mathfrak{a}}_+ = \coprod_{n>0} \tilde{\mathfrak{a}}_+^{\phi_n}$. We have just proven

Proposition 3.1.2: $\tilde{\mathfrak{a}}$ is a Lie algebra with triangular decomposition.

§3.2 The $\tilde{\mathfrak{a}}$-module $V(\lambda)$, $\lambda\in\Gamma$

Let $\Gamma = \dot{Q}\oplus\mathbb{Z}\delta\oplus\mathbb{Z}\mu = Q\oplus\mathbb{Z}\mu$ be the nondegenerate lattice with $(\dot{Q}|\delta) = 0 = (\mu|\dot{Q})$ and $(\mu|\delta) = 1 = (\delta|\mu)$. Thus $\Gamma = \dot{Q}\oplus\Lambda$ and $(\dot{Q}|\Lambda) = 0$ so we write $\Gamma = \dot{Q}\perp\Lambda$. Fix $\lambda\in\Gamma$ and let $V(\lambda) \equiv V_\Gamma(\lambda) := \mathbb{C}e^\lambda\otimes_{\mathbb{C}}S(\mathfrak{a}(\Gamma)_-)$ be the Fock space corresponding to Γ. Thus $V(\lambda)$ is the canonical representation of $\mathfrak{a}(\Gamma)$ and since Γ is nondegenerate $V(\lambda)$ is an irreducible $\mathfrak{a}(\Gamma)$-module.

The objective of this section is to explain how $V(\lambda)$ is a module for the semi-direct product algebra $\tilde{\mathfrak{a}}$ constructed in §1.

Let $\{a_i\}_{i=1}^{\ell}$ be an orthonormal basis for $\dot{\mathfrak{h}} := \mathbb{C}\otimes_{\mathbb{Z}}\dot{Q}$. Since $\dot{Q}$ is a positive definite lattice this can be done over $\mathbb{R}$. Let $\{\alpha_1,\alpha_2\}$ be the orthonormal basis for $\mathfrak{p} = \mathbb{C}\otimes_{\mathbb{Z}}\Lambda$ used in chapter 2. Now let $u_i := a_i$, $i=1,\dots,\ell$ and $u_{\ell+1} := \alpha_1$ and $u_{\ell+2} := \alpha_2$. Then $\{u_i\}_{i=1}^{\ell+2}$ is an orthonormal basis over $\mathbb{C}$ for $\mathfrak{k} := \mathbb{C}\otimes_{\mathbb{Z}}\Gamma$. Thus by proposition (2.2.1) the operators

$$L_k := \frac{1}{2}\sum_{j\in\mathbb{Z}}\sum_{i=1}^{\ell+2}:u_i(-j)u_i(j+k):$$ on $V_\Gamma(\lambda)$ define a representation of $\mathcal{V}ir$ where the centre $\mathfrak{z}$ acts as $(\ell+2)I$.

Fix $\lambda\in\Gamma$ and write $\lambda=\alpha+\beta$ where $\alpha\in\dot{Q}$ and $\beta\in\Lambda$. Consider the representation

$$d_k \longmapsto \dot{L}_k := \frac{1}{2}\sum_{j\in\mathbb{Z}}\sum_{i=1}^{\ell}:u_i(-j)u_i(j+k):,\quad \mathfrak{z}\longmapsto \ell I$$ of $\mathcal{V}ir$ on $V_{\dot{Q}}(\alpha)$ and the representation of $\mathcal{V}ir$ on $V_\Lambda(\beta)$ given in chapter 2 by $d_k\longmapsto H_k$ and $\mathfrak{z}\longmapsto 2I$. Then we have

Proposition 3.2.1: $V_\Gamma(\lambda) \cong V_{\dot{Q}}(\alpha) \otimes_{\mathbb{C}} V_\Lambda(\beta)$ as modules over $\mathcal{V}ir$.

Proof: Indeed, this follows from the fact that $\dot{Q} \perp \Lambda$. We omit the details here since the proof is entirely analogous to the one in proposition 2.3.1. We will, however, point out that the linear isomorphism

$\phi : V_{\dot{Q}}(\alpha) \otimes_{\mathbb{C}} V_\Lambda(\beta) \longrightarrow V(\lambda) : (e^{\alpha}\otimes f) \otimes (e^{\beta}\otimes g) \longmapsto e^{\alpha+\beta}\otimes fg$ is a $\mathcal{V}ir$-module map and under this correspondence the operator L_k on $V_\Gamma(\lambda)$ corresponds to the operator $(\dot{L}_k\otimes I)+(I\otimes H_k)$ on $V_{\dot{Q}}(\alpha) \otimes_{\mathbb{C}} V_\Lambda(\beta)$. ■

Proposition 3.2.2: $V(\lambda) \equiv V_\Gamma(\lambda)$ is an $\tilde{\mathfrak{a}}$-module.

Proof: We know that $V(\lambda)$ is a module for both $\mathcal{V}ir$ and $\mathfrak{a}(Q)$ individually. However, from §2.2 we have the commutation relations:

$[L_k,a(n)] = -na(n+k)$, $n,k\in\mathbb{Z}$ and $a\in Q$. Now since $[d_k,a(n)] = -na(n+k)$, the result is clear by virtue of definition 3.1.2. ■

§ 3.3 The structure of $V(\lambda)$, $\lambda \in \Gamma \backslash Q$.

In this section we will see that the semi-direct product algebra $\tilde{\mathfrak{a}}$ is an example of the adage, "the whole is greater than the sum of its parts". Our goal is to show that, in the case $\lambda \in \Gamma \backslash Q$, the space $V(\lambda) \equiv V_\Gamma(\lambda)$, while complicated as both an $\mathfrak{a}$-module and as a $\mathcal{V}ir$-module, is irreducible as an $\tilde{\mathfrak{a}}$-module.

Recall $\mathcal{V}ir_+ = \coprod_{n>0} \mathbb{C}d_n$. Introduce the subalgebra $\hat{\mathfrak{a}} \equiv \hat{\mathfrak{a}}(Q) := \mathcal{V}ir_+ \propto \mathfrak{a}$ of $\tilde{\mathfrak{a}}$. Then we have the following inclusion of algebras:

$$\mathfrak{a} \subset \hat{\mathfrak{a}} \subset \tilde{\mathfrak{a}}.$$

Proposition 3.3.1: Fix $\lambda \in \Gamma$ arbitrarily. Then $V_Q(\lambda) := \mathbb{C}e^\lambda \otimes S(\mathfrak{a}(Q)_-)$ is an $\hat{\mathfrak{a}}$-submodule of $V(\lambda)$.

Proof: Clearly $V_Q(\lambda)$ is $\mathfrak{a}(Q)$-invariant (§1.4). Also, from (3.1.2), we know that $\mathcal{V}ir$ acts on $\mathfrak{a}(Q)$. Extend this latter action to an action of $\mathcal{V}ir$ on $S(\mathfrak{a}(Q))$ in such a way that for $k \in \mathbb{Z}$, d_k is the unique derivation of $S(\mathfrak{a}(Q))$ satisfying (3.1.2) and such that $\mathfrak{z}$ acts trivially. Then, for any homogeneous polynomial $f \in S(\mathfrak{a}(Q)_-)$, it is clear that $d_0(f) = (\deg f)f$. To prove the proposition, it suffices to show that for any $n \geq 0$,

$$d_n \cdot (e^\lambda \otimes f) = \left(\delta_{n,0} \frac{(\lambda|\lambda)}{2} f + d_n(f)\right) \cdot (e^\lambda \otimes 1) \tag{3.3.1}$$

since the right side of this equation belongs to $V_Q(\lambda)$. Consider $n=0$ first.

$$d_0 \cdot (e^{\lambda}\otimes f) = \frac{1}{2} \sum_{j\in\mathbb{Z}} \sum_{i=1}^{\ell+2} :u_i(-j)u_i(j): \ (e^{\lambda}\otimes f)$$

$$= \frac{1}{2} \sum_{i=1}^{\ell+2} u_i(0)u_i(0) \ (e^{\lambda}\otimes f) + \sum_{j>0} \sum_{i=1}^{\ell+2} u_i(-j)u_i(j) \ (e^{\lambda}\otimes f)$$

$$= \left(\frac{(\lambda|\lambda)}{2} + \text{degf}\right)(e^{\lambda}\otimes f)$$

$$= \left(\frac{(\lambda|\lambda)}{2} f + d_0(f)\right) \ (e^{\lambda}\otimes 1).$$

As for $n>0$, first note that $d_n\cdot(e^{\lambda}\otimes 1) = 0$ since every summand $:u_i(-j)u_i(j+n):$ appearing in the definition of L_n can be written $u_i(p)u_i(q)$ where $p\le 0$, $q>0$ (after removing the normal ordering). However, each of these terms kills $e^{\lambda}\otimes 1$. Thus $d_n\cdot(e^{\lambda}\otimes f) = d_n\cdot f\cdot(e^{\lambda}\otimes 1)$

$$= f\cdot d_n\cdot(e^{\lambda}\otimes 1) + d_n(f)\cdot(e^{\lambda}\otimes 1)$$

$$= d_n(f)\cdot(e^{\lambda}\otimes 1) \quad \text{as required.} \qquad \blacksquare$$

Before proceeding to the main result of this section we will need a preliminary definition. For $k>0$ recall the set $\mathcal{P}_k$ of all partitions of k (§2.8). We know that there is a one-to-one correspondence between the elements $\mathbf{m} = (m_1,\dots,m_r) \in \mathbb{Z}_+^r$ with $m_1>m_2>\cdots>m_r$ and $\sum_{i=1}^{r} m_i = k$ and the monomials $\delta(-\mathbf{m}) = \delta(-m_1)\cdots\delta(-m_r)$ of degree k. Now we give $\mathcal{P}_k$ the lexicographical ordering as follows. For $\mathbf{m} = (m_1,\dots,m_r)$, $\mathbf{n} = (n_1,\dots,n_s)\in\mathcal{P}_k$, $r,s>0$ we say $\mathbf{m}<\mathbf{n}$ if $m_i<n_i$ for the first i such that $m_i\neq n_i$. Clearly then the

partition $(k) \in \mathcal{P}_k$ of length 1 is the unique maximal element with respect to this ordering and the partition $(1,\dots,1) \in \mathcal{P}_k$ of length k is the unique minimal element. In the next three results, we assume that all given tuples $\mathbf{m} = (m_1,\dots,m_r) \in \mathbb{Z}_+^r$, $r>0$ are ordered in such a way that $m_1>m_2>\cdots>m_r$.

Lemma 3.3.2: Fix $\lambda\in\Gamma\backslash Q$ and suppose $\mathbf{m}\in\mathbb{Z}_+^r$, $\mathbf{n}\in\mathbb{Z}_+^s$ where $\mathbf{n},\mathbf{m}\in\mathcal{P}_k$ and $\mathbf{n}<\mathbf{m}$ in the lexicographical ordering. Consider the elements $e^{\lambda}\otimes\delta(-\mathbf{m})$, $e^{\lambda}\otimes\delta(-\mathbf{n}) \in V_Q(\lambda)$. Then $d_{\mathbf{m}}\cdot(e^{\lambda}\otimes\delta(-\mathbf{m})) \in \mathbb{C}^{\times}$ and $d_{\mathbf{m}}\cdot(e^{\lambda}\otimes\delta(-\mathbf{n})) = 0$ where

$d_{\mathbf{m}} := d_{m_r}\cdots d_{m_1} \in \mathcal{U}(\mathcal{V}ir_+)$.

Proof: $d_{\mathbf{m}}\cdot(e^{\lambda}\otimes\delta(-\mathbf{m})) = d_{m_r}\cdots d_{m_1}\cdot(e^{\lambda}\otimes\delta(-m_1)\cdots\delta(-m_r))$

$= d_{m_t}^{p_t}\cdots d_{m_1}^{p_1}\cdot(e^{\lambda}\otimes\delta(-m_1)^{p_1}\cdots\delta(-m_t)^{p_t})$ where $m_i\neq m_j$, $i\neq j$ and $p_k>0$, $1\leq k\leq t$

$= \left(p_1!(m_1)^{p_1}(\lambda|\delta)^{p_1}\right) d_{m_t}^{p_t}\cdots d_{m_2}^{p_2}\cdot\left(e^{\lambda}\otimes\delta(-m_2)^{p_2}\cdots\delta(-m_t)^{p_t}\right)$, by (3.3.1)

$= \cdots = \left((p_1!\cdots p_t!) \prod_{i=1}^{t} m_i^{p_i} (\lambda|\delta)^{p_1\cdots p_t} \right) (e^{\lambda}\otimes 1) \in \mathbb{C}^{\times}$ since $(\lambda|\delta)\neq 0$.

As for the second part, suppose $m_i=n_i$, $1\leq i<i_0$ but $m_{i_0}>n_{i_0}$. Then, in the calculation of $d_{\mathbf{m}}\cdot(e^{\lambda}\otimes\delta(-\mathbf{n}))$, the same computation as above gives

$d_{\mathbf{m}}\cdot(e^{\lambda}\otimes\delta(-\mathbf{n})) = K\, d_{m_t}^{p_t}\cdots d_{m_{i_0}}^{p_{i_0}}\cdot\left(e^{\lambda}\otimes\delta(-n_{i_0})^{q_{i_0}}\cdots\delta(-n_u)^{q_u}\right)$, $K\in\mathbb{C}^{\times}$, $q_j>0$, $i_0\leq j\leq u$.

However, since $m_{i_0}>n_{i_0}$ the next step of this calculation (when we apply $d_{m_{i_0}}$) produces zero. ∎

Proposition 3.3.3: Let $\lambda\in\Gamma\backslash Q$. Then $V_Q(\lambda) = \mathbb{C}e^{\lambda}\otimes_{\mathbb{C}} S(\mathfrak{a}(Q)_-)$ is an irreducible $\hat{\mathfrak{a}}$-module.

Proof: First note that $\lambda\in\Gamma\backslash Q$ is equivalent to the condition $(\lambda|\delta)\neq 0$.

Let W be a nonzero $\hat{\mathfrak{a}}$-submodule of $V_Q(\lambda)$ and let $0 \neq x \in W$. Write $x = e^{\lambda}\otimes \sum_{i=1}^{m} g_i h_i$, $g_i\in S(\mathfrak{a}(\dot{Q})_-)$ and $h_i\in D = S\left(\sum_{m>0}\mathbb{C}\delta(-m)\right)$. By arguing as in the proof of proposition 1.4.2 (ii)$\Rightarrow$(iii), we can clear the $S(\mathfrak{a}(\dot{Q})_-)$ part and assume without loss of generality that $x = e^{\lambda}\otimes h$. Write $h = \sum_{i=1}^{r}\alpha_i\delta(-\mathbf{m}_i)$, $\alpha_i\in\mathbb{C}^{\times}$ where $\mathbf{m}_1>\mathbf{m}_2>\cdots>\mathbf{m}_r$ in the lexicographical ordering. Finally, since the elements d_n, $n>0$ act as homogeneous operators of degree $-n$ it is clear from lemma 3.3.2 that $d_{\mathbf{m}_1}\cdot(e^{\lambda}\otimes h) \in \mathbb{C}^{\times}$. ∎

Proposition 3.3.4: Let $\lambda\in\Gamma\backslash Q$. Then $V(\lambda) \equiv V_{\Gamma}(\lambda) = \mathbb{C}e^{\lambda}\otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma)_-)$ is an irreducible $\tilde{\mathfrak{a}}$-module.

Proof: Let W be a nonzero $\tilde{\mathfrak{a}}$-submodule of $V(\lambda)$. Let $0\neq f\in W$ be arbitrary and write $f = \sum_{i=1}^{q} e^{\lambda}\otimes f_i$, $f_i\in S(\mathfrak{a}(\Gamma)_-)$. Since $\Gamma = Q\oplus\mathbb{Z}\mu$ we can write

$$S(\mathfrak{a}(\Gamma)_-) = S(\mathfrak{a}(Q)_-)S\left(\coprod_{n>0}\mathbb{C}\mu(-n)\right) = S(\mathfrak{a}(Q)_-)M \quad \text{where } M := S\left(\coprod_{n>0}\mathbb{C}\mu(-n)\right).$$

Write $f_i = h_i k_i$ for $i = 1,\ldots,q$ where $h_i\in S(\mathfrak{a}(Q)_-)$ and $k_i\in M$. Again, using an argument as in proposition 1.4.2 (ii)$\Rightarrow$(iii), we can use the $\delta(n)$, $n>0$ and reduce to the case where $f = e^{\lambda}\otimes h \in W$ where $h \in S(\mathfrak{a}(Q)_-)$. But by proposition 3.3.3, $V_Q(\lambda)$ is an irreducible $\hat{\mathfrak{a}}$-module. Thus $e^{\lambda}\otimes 1 \in W$ and it suffices to show

$$N := \mathcal{U}(\tilde{\mathfrak{a}})\cdot(e^{\lambda}\otimes 1) \supset V(\lambda) = \mathbb{C}e^{\lambda}\otimes_{\mathbb{C}} S(\mathfrak{a}(Q)_-)M. \tag{3.3.2}$$

Now write $M = \coprod_{n\geq 0} M_n$ where M_n denotes the subspace of M spanned by elements of

degree n. We will establish (3.3.2) by induction on the degree n in this decomposition. Since $M_0 = \mathbb{C}$ and $M_1 = \mathbb{C}\mu(-1)$, we can write $M = \mathbb{C} \oplus \mathbb{C}\mu(-1) \oplus \left(\coprod_{n=2}^{\infty} M_n\right)$. Since $\mathfrak{U}(\tilde{\mathfrak{a}})\cdot(e^{\lambda}\otimes 1) \supset \mathfrak{U}(\mathfrak{a}(Q))\cdot(e^{\lambda}\otimes 1) = \mathbb{C}e^{\lambda}\otimes_{\mathbb{C}} S(\mathfrak{a}(Q)_-)$, (3.3.2) is clear when n=0. For n=1 consider $d_{-1}\cdot(e^{\lambda}\otimes 1) = L_{-1}(e^{\lambda}\otimes 1) \in N$. By definition we have

$$d_{-1}\cdot(e^{\lambda}\otimes 1) = \sum_{i=1}^{\ell+2} u_i(-1)u_i(0) \cdot (e^{\lambda}\otimes 1)$$

$$= \sum_{i=1}^{\ell+2} u_i(-1)(\lambda\,|\,u_i)\cdot(e^{\lambda}\otimes 1)$$

$$= e^{\lambda}\otimes \left(\sum_{i=1}^{\ell+2} (\lambda\,|\,u_i)u_i(-1)\right)$$

$$= e^{\lambda}\otimes \lambda(-1).$$

Now writing $\lambda = \alpha + a\mu$, $\alpha\in Q$, $a\in\mathbb{C}^{\times}$ we have $L_{-1}(e^{\lambda}\otimes 1) = e^{\lambda}\otimes\alpha(-1) + a(e^{\lambda}\otimes\mu(-1)) \in N$ But $e^{\lambda}\otimes\alpha(-1) \in N$ from the case n=0. Thus we have $e^{\lambda}\otimes\mu(-1) \in N$ and this shows (3.3.2) when n=1.

Next suppose $N \supset \mathbb{C}e^{\lambda}\otimes_{\mathbb{C}}\left(\coprod_{0\le n\le k-1} M_n S(\mathfrak{a}(Q)_-)\right)$. We call this the first induction hypothesis. We need to show $N \supset \mathbb{C}e^{\lambda}\otimes_{\mathbb{C}} M_k S(\mathfrak{a}(Q)_-)$. We prove this by induction on the lexicographical ordering defined on $\mathcal{P}_k$. We "anchor" at the top with the partition (k). That is, we will first show $e^{\lambda}\otimes\mu(-k) \in N$.

Recall that for k>0, $L_{-k} = \frac{1}{2} \sum_{j\in\mathbb{Z}} \sum_{i=1}^{\ell+2} :u_i(-j)u_i(j-k):$

$$= \frac{1}{2} \sum_{j\in\mathbb{Z}} \left\{ \sum_{i=1}^{\ell} :u_i(-j)u_i(j-k): + :\delta(-j)\mu(j-k): + :\mu(-j)\delta(j-k): \right\}.$$

Note that in the expansion of $L_{-k}(e^{\lambda}\otimes 1) \in N$ the only $j\in\mathbb{Z}$ which contribute are $j = k,k-1,\ldots,\left[\frac{k+1}{2}\right]$ where [x] denotes the largest integer less than x.

We compute $L_{-k}(e^{\lambda}\otimes 1) = \frac{1}{2}\left(\sum_{i=1}^{\ell+2} u_i(-k)u_i(0)\right)(e^{\lambda}\otimes 1)$

$$+ \left\{ \left(\frac{1}{2}\sum_{i=1}^{\ell} u_i(-k+1)u_i(-1) + \frac{1}{2}\,\delta(-k+1)\mu(-1) + \frac{1}{2}\,\mu(-k+1)\delta(-1) \right) + \cdots \right.$$

$$+ \frac{1}{2} \sum_{i=1}^{\ell} u_i\left(-\left[\frac{k+1}{2}\right]\right)\, u_i\left(\left[\frac{k+1}{2}\right]-k\right)$$

$$\left. + \frac{1}{2}\,\delta\left(-\left[\frac{k+1}{2}\right]\right)\,\mu\left(\left[\frac{k+1}{2}\right]-k\right) + \mu\left(-\left[\frac{k+1}{2}\right]\right)\,\delta\left(\left[\frac{k+1}{2}\right]-k\right) \right\} (e^{\lambda}\otimes 1)$$

$= \frac{1}{2}\, e^{\lambda}\otimes\lambda(-k) + x$ where $x \in \mathbb{C}e^{\lambda}\otimes_{\mathbb{C}}\left(\coprod_{n<k} M_n S(a(Q)_-) \right)$ and $x\in N$ by the first induction hypothesis. Thus $e^{\lambda}\otimes\mu(-k) \in N$ by writing $\lambda = \alpha + a\mu$ and arguing as we did earlier in the case n=1.

Next we fix $\mathbf{m} = (m_1,\ldots,m_r) \in \mathcal{P}_k$ and assume that for every $n\in\mathcal{P}_k$

satisfying $n>m$ we have $e^{\lambda}\otimes\mu(-n) \in N$. We call this the second induction hypothesis. We need to show that $e^{\lambda}\otimes\mu(-m) \in N$. Indeed, since $m_1>0$, $\sum_{i=2}^{r} m_i<k$ and so by the first induction hypothesis $e^{\lambda}\otimes\mu(-m_2)\cdots\mu(-m_r) \in N$. But then N also contains the element $x := L_{-m_1}\left(e^{\lambda}\otimes\mu(-m_2)\cdots\mu(-m_r)\right)$ Now, in the sum defining L_{-m_1}, the only $j\in\mathbb{Z}$ which contribute in the calculation of x are

$j\in \{m_1\} \cup \left\{ m_1-1,\ldots,\left[\frac{m_1+1}{2}\right] \right\} \cup \{m_1+m_2,\ldots,m_1+m_r\}$. We calculate

$$L_{-m_1}\left(e^{\lambda}\otimes\mu(-m_2)\cdots\mu(-m_r)\right) = \frac{1}{2}\left(e^{\lambda}\otimes\lambda(-m_1)\mu(-m_2)\cdots\mu(-m_r)\right)$$

$$+ \left\{ \frac{1}{2}\sum_{i=1}^{\ell} u_i(-m_1+1)u_i(-1) + \frac{1}{2}\,\delta(-m_1+1)\mu(-1) + \frac{1}{2}\,\mu(-m_1+1)\delta(-1) + \cdots \right.$$

$$+ \frac{1}{2}\sum_{i=1}^{\ell} u_i\left(-\left[\frac{m_1+1}{2}\right]\right)\, u_i\left(\left[\frac{m_1+1}{2}\right]-m_1\right) +$$

$$\left.\frac{1}{2}\,\delta\left(-\left[\frac{m_1+1}{2}\right]\right)\,\mu\left(\left[\frac{m_1+1}{2}\right]-m_1\right) + \frac{1}{2}\,\mu\left(-\left[\frac{m_1+1}{2}\right]\right)\,\delta\left(\left[\frac{m_1+1}{2}\right]-m_1\right)\right\}\cdot(e^{\lambda}\otimes 1)$$

$$+ \sum_{i=1}^{r} \frac{m_i}{2}\left(e^{\lambda}\otimes\mu(-m_1-m_1)\mu(-m_2)\cdots\overline{\mu(-m_i)}\cdots\mu(-m_r)\right)$$ where the overbar denotes omission.

Let x_1 denote the sum in the bracebrackets and x_2 the sum with the overbars. By the first induction hypothesis $x_1\cdot(e^{\lambda}\otimes 1) \in N$ and since $(m_1+m_i\ ,\ m_2\ ,\ldots,\ \overline{m_i}\ ,\ldots,\ m_r) > (m_2\ ,\ m_3\ ,\ldots,\ m_r)$ for each $2\le i\le r$, the second

induction hypothesis implies $x_2 \in N$. Finally since the left side belongs to N we conclude that $e^{\lambda}\otimes\lambda(-m_1)\mu(-m_2)\cdots\mu(-m_r) \in N$. Expressing $\lambda = \alpha + a\mu$, $\alpha\in Q$, $a\in\mathbb{C}^{\times}$ the first induction hypothesis gives $e^{\lambda}\otimes\mu(-m_1)\cdots\mu(-m_r) \in N$ as required. This completes the proof of proposition 3.3.4. ∎

Our next goal is to identify $V(\lambda)$, $\lambda\in\Gamma\setminus Q$ as an irreducible highest weight module. Recall from §3.1 that $\tilde{\mathfrak{a}}$ admits a triangular decomposition: $\tilde{\mathfrak{a}} = \tilde{\mathfrak{a}}_- \oplus \tilde{\mathfrak{a}}_0 \oplus \tilde{\mathfrak{a}}_+$. Let $\alpha \in (\tilde{\mathfrak{a}}_0)^*$ be defined by

$$\begin{cases} \alpha(a(0)) := (\lambda|a) \\ \alpha(c) := 1 \\ \alpha(d_0) := \dfrac{(\lambda|\lambda)}{2} \\ \alpha(\mathfrak{z}) := \ell+2 \end{cases}$$

and consider the Verma module $M(\alpha) = \mathfrak{U}(\tilde{\mathfrak{a}})\otimes_{\mathfrak{U}(\mathfrak{b})}\mathbb{C}_{\alpha}$, $\mathfrak{b} = \tilde{\mathfrak{a}}_0 \oplus \tilde{\mathfrak{a}}_+$ corresponding to α. Now let $L(\alpha)$ be the unique irreducible quotient by its unique maximal proper submodule $N(\alpha)$. Then we have

Proposition 3.3.5: $V(\lambda) \cong L(\alpha)$

Proof: Let $f : \mathbb{C}_{\alpha} \longrightarrow V(\lambda)$ be the unique $\mathbb{C}$-linear map satisfying $f(v_+) := e^{\lambda}\otimes 1$. It is easy to check that f is a $\mathfrak{b}$-module map. By the universal property of the induced module, there exists a unique $\mathfrak{U}(\tilde{\mathfrak{a}})$-module map $\phi : M(\alpha) \longrightarrow V(\lambda)$ such that $\phi \circ \iota = f$ where $\iota : \mathbb{C}_{\alpha} \longrightarrow M(\alpha)$ is the natural inclusion. Thus if $u \in \mathfrak{U}(\tilde{\mathfrak{a}})$ then $\phi(u\otimes v_+) = u\cdot(e^{\lambda}\otimes 1)$ and since $V(\lambda)$

is an irreducible highest weight module we have $V(\lambda) \cong M(\alpha)/N(\alpha) = L(\alpha)$ where $N(\alpha) = \ker\phi$.

Remark: The question of whether $V(\lambda)$ is a Verma module, remains unanswered. This question is, of course, equivalent to the problem of finding the kernel of the map ϕ above. Since the maximum proper submodule $N(\alpha)$ of $M(\alpha)$ is the radical of the Shapavalov form $(\cdot|\cdot)$ on $M(\alpha)$, if one could obtain a formula for the value of the form it might be possible to determine from it whether or not $V(\lambda)$ is a Verma module.

§3.4 The structure of $V(\lambda)$, $\lambda \in Q$

In this section we will investigate the structure of the $\tilde{\mathfrak{a}}$-module $V(\lambda)$ when $\lambda \in Q$. Unless otherwise stated, we will fix $\lambda := \alpha + n\delta \in Q$ arbitrarily where $\alpha \in \dot{Q}$ and $n \in \mathbb{Z}$. Often we will denote $V(\lambda)$ simply by V.

Our strategy will be as follows. Recall $V_{\dot{Q}}(\alpha) = \mathbb{C}e^{\alpha} \otimes_{\mathbb{C}} S(\mathfrak{a}(\dot{Q})_-)$ and $V_{\Lambda}(n\delta) \equiv W(n\delta) = \mathbb{C}e^{n\delta} \otimes_{\mathbb{C}} S(\mathfrak{a}(\Lambda)_-)$. Consider the $\mathbb{C}$-space $V_{\dot{Q}}(\alpha) \otimes_{\mathbb{C}} W(n\delta)$. This space is isomorphic to $V(\lambda)$ and we will use the isomorphism to transfer the $\tilde{\mathfrak{a}}$-module structure from $V(\lambda)$ to it. Once this is done, we can use the $\mathcal{V}ir$-submodules of $W(n\delta)$ to produce $\tilde{\mathfrak{a}}$-submodules of $V(\lambda)$.

Define $\phi : V_{\dot{Q}}(\alpha) \otimes_{\mathbb{C}} W(n\delta) \longrightarrow V(\lambda)$ to be the unique $\mathbb{C}$-linear map satisfying

$$\phi\left[(e^{\alpha} \otimes f) \otimes (e^{n\delta} \otimes g)\right] := e^{\lambda} \otimes fg \tag{3.4.1}$$

where $f \in S(\mathfrak{a}(\dot{Q})_-)$ and $g \in S(\mathfrak{a}(\Lambda)_-)$. It is clear that ϕ is a linear isomorphism and by proposition 3.2.1 we have

Proposition 3.4.1: $V_{\dot{Q}}(\alpha) \otimes_{\mathbb{C}} W(n\delta)$ becomes an $\tilde{\mathfrak{a}}$-module, isomorphic to $V(\lambda)$ under ϕ, in such a way that

$$\begin{cases} a(m) \cdot \left[(e^{\alpha} \otimes f) \otimes (e^{n\delta} \otimes g)\right] = \left[a(m) \cdot e^{\alpha} \otimes f\right) \otimes (e^{n\delta} \otimes g) \\ \delta(m) \cdot \left[(e^{\alpha} \otimes f) \otimes (e^{n\delta} \otimes g)\right] = (e^{\alpha} \otimes f) \otimes \left[\delta(m) \cdot (e^{n\delta} \otimes g)\right] \\ d_m \cdot \left[(e^{\alpha} \otimes f) \otimes (e^{n\delta} \otimes g)\right] = (\dot{L}_m \otimes I) + (I \otimes H_m)\left[(e^{\alpha} \otimes f) \otimes (e^{n\delta} \otimes g)\right] \end{cases} \tag{3.4.2}$$

where $a \in \dot{Q}$, $m \in \mathbb{Z}$ and we have used the fact that $V_{\dot{Q}}^{\bullet}(\alpha)$ is an $\mathfrak{a}(\dot{Q})$-module and $W(n\delta)$ is an $\mathfrak{a}(\mathbb{Z}\delta)$-module.

Recall from chapter 2 that for all $k \in \mathbb{Z}$, $W^k(n\delta) = \mathbb{C}e^{n\delta} \otimes_{\mathbb{C}} \left(\coprod_{j \geq 0} \Omega_{\leq (j+k)}{}^{\Delta_j} \right) \subset W(n\delta)$ and $W^k(n\delta) \subset W^{k+1}(n\delta)$. Note that $W^k(n\delta)$ is an $\mathfrak{a}(\mathbb{Z}\delta)$-submodule of $W(n\delta)$ for all $k \in \mathbb{Z}$. Moreover, by proposition 2.4.2(i) we know that $W^k(n\delta)$ is a $\mathcal{V}ir$-submodule of $W(n\delta)$. Thus by (3.4.2), for every $k \in \mathbb{Z}$, $V_{\dot{Q}}^{\bullet}(\alpha) \otimes_{\mathbb{C}} W^k(n\delta)$ is an $\tilde{\mathfrak{a}}$-submodule of $V_{\dot{Q}}^{\bullet}(\alpha) \otimes_{\mathbb{C}} W(n\delta)$. Next let $\phi^k := \phi \big|_{V_{\dot{Q}}^{\bullet}(\alpha) \otimes_{\mathbb{C}} W^k(n\delta)}$ and put $V^k := \operatorname{Im} \phi^k$. Then since ϕ is an $\tilde{\mathfrak{a}}$-module map we have

Proposition 3.4.2: The set $\{V^k\}_{k \in \mathbb{Z}}$ is a filtration of $\tilde{\mathfrak{a}}$-submodules of V satisfying $V^k \subset V^{k+1}$, $\forall k \in \mathbb{Z}$.

For every $k \in \mathbb{Z}$ define $V_k := V^k / V^{k-1}$. Our next goal is to express the $\tilde{\mathfrak{a}}$-module structure of V_k in terms of the $\mathcal{V}ir$-module structure of $W_k(n\delta)$.

Recall from chapter 2 that $W_k(n\delta) = W^k(n\delta)/W^{k-1}(n\delta)$ is a completely reducible $\mathcal{V}ir$-module with $W_k(n\delta) = \coprod_{\gamma \in I_k} W_k^{\gamma}(n\delta)$ where $W_k^{\gamma}(n\delta)$ is an irreducible highest weight module generated by the highest weight vector $\overline{e^{n\delta} \otimes \xi_k^{\gamma}}$. Now, as $\mathbb{C}$-spaces,

$$V_{\dot{Q}}^{\bullet}(\alpha) \otimes_{\mathbb{C}} W^k(n\delta) \Big/ V_{\dot{Q}}^{\bullet}(\alpha) \otimes_{\mathbb{C}} W^{k-1}(n\delta) \overset{\nu_k}{\cong} V_{\dot{Q}}^{\bullet}(\alpha) \otimes_{\mathbb{C}} W_k(n\delta) \tag{3.4.3}$$

and if we transfer the $\tilde{\mathfrak{a}}$-module structure from the left side of (3.4.3) to the

right side it becomes

$$(3.4.4)\quad \begin{cases} a(m)\cdot\left((e^{\alpha}\otimes f)\otimes\overline{(e^{n\delta}\otimes g)}\right) = \left(a(m)\cdot(e^{\alpha}\otimes f)\right)\otimes\overline{(e^{n\delta}\otimes g)} \\ \delta(m)\cdot\left((e^{\alpha}\otimes f)\otimes\overline{(e^{n\delta}\otimes g)}\right) = 0 \\ d_m\cdot\left((e^{\alpha}\otimes f)\otimes\overline{(e^{n\delta}\otimes g)}\right) = \left(\dot{L}_m(e^{\alpha}\otimes f)\otimes\overline{(e^{n\delta}\otimes g)}\right) + \left((e^{\alpha}\otimes f)\otimes H_m\overline{(e^{n\delta}\otimes g)}\right) \end{cases}$$

where $f \in S(\mathfrak{a}(\dot{Q})_-)$ and $g \in \Theta^k$.

Now define $\phi_k : V_{\dot{Q}}(\alpha)\otimes_{\mathbb{C}} W_k(n\delta) \longrightarrow V_k$ to be the unique linear map satisfying

$$(3.4.5)\qquad \phi_k\left((e^{\alpha}\otimes f)\otimes\overline{(e^{n\delta}\otimes g)}\right) := \overline{e^{\lambda}\otimes fg}$$

where $f \in S(\mathfrak{a}(\dot{Q})_-)$ and $g \in \Theta^k$. Then ϕ_k is a $\tilde{\mathfrak{a}}$-module map since it is a composition of two such maps: $\phi_k = (\bar{\phi}^k)\circ\nu_k^{-1}$ where $\bar{\phi}^k : V_Q^{\bullet}(\alpha)\otimes_{\mathbb{C}} W^k(n\delta) \,/\, V_Q^{\bullet}(\alpha)\otimes_{\mathbb{C}} W^{k-1}(n\delta) \longrightarrow V^k/V^{k-1}$ is the map $\bar{\phi}^k(\bar{x}) := \overline{\phi^k(x)}$, $x \in V_Q^{\bullet}(\alpha)\otimes_{\mathbb{C}} W^k(n\delta)$.

Next for $k\in\mathbb{Z}$ and $\gamma\in I_k$ define the following cyclic submodules of V_k:

$$(3.4.6)\qquad V_k^{\gamma} := \mathfrak{U}(\tilde{\mathfrak{a}})\cdot\overline{(e^{\lambda}\otimes\xi_k^{\gamma})}.$$

Lemma 3.4.3: Let $\overline{e^{\lambda}\otimes g} \in V_k$, $g \in \Theta^k$. Let $m\in\mathbb{Z}$ and set $x := H_m(\overline{e^{n\delta}\otimes g}) \in W_k(n\delta)$. Then $\phi_k\left((e^{\alpha}\otimes 1)\otimes x\right) \in \mathcal{U}(\tilde{\mathfrak{a}})\cdot(\overline{e^{\lambda}\otimes g})$.

Proof: By (3.4.4),

$$\phi_k\left\{\left(\dot{L}_m(e^{\alpha}\otimes 1)\otimes(\overline{e^{n\delta}\otimes g})\right) + \left((e^{\alpha}\otimes 1)\otimes H_m(\overline{e^{n\delta}\otimes g})\right)\right\} = d_k\cdot(\overline{e^{\lambda}\otimes g}) \in \mathcal{U}(\tilde{\mathfrak{a}})\cdot(\overline{e^{\lambda}\otimes g}).$$

However, $\phi_k\left(\dot{L}_m(e^{\alpha}\otimes 1)\otimes(\overline{e^{n\delta}\otimes g})\right) \in \mathcal{U}(\mathfrak{a}(\dot{Q}))\cdot(\overline{e^{\lambda}\otimes g}) \subset \mathcal{U}(\tilde{\mathfrak{a}})\cdot(\overline{e^{\lambda}\otimes g})$.
Now subtracting, we have the desired result. ∎

Proposition 3.4.4: (i) $V_k^{\gamma} = \phi_k\left(V_{\dot{Q}}(\alpha)\otimes_{\mathbb{C}} W_k^{\gamma}(n\delta)\right)$ and consequently, $V_k = \coprod_{\gamma\in I_k} V_k^{\gamma}$.

(ii) V_k^{γ} is an irreducible $\tilde{\mathfrak{a}}$-module and consequently, V_k is completely reducible.

Proof: For (i), it suffices to show that $V_k^{\gamma} \supset \phi_k\left(V_{\dot{Q}}(\alpha)\otimes_{\mathbb{C}} W_k^{\gamma}(n\delta)\right)$. Indeed,

by lemma 3.4.3, V_k^{γ} contains $\phi_k\left(\mathbb{C}(e^{\alpha}\otimes 1)\otimes_{\mathbb{C}} W_k^{\gamma}(n\delta)\right)$ and hence $V_k^{\gamma} \supset \mathcal{U}(\mathfrak{a}(\dot{Q}))\cdot\phi_k\left(\mathbb{C}(e^{\alpha}\otimes 1)\otimes_{\mathbb{C}} W_k^{\gamma}(n\delta)\right) = \phi_k\left(V_{\dot{Q}}(\alpha)\otimes_{\mathbb{C}} W_k^{\gamma}(n\delta)\right)$ since ϕ_k is an $\tilde{\mathfrak{a}}$-module map. Finally, for (ii) we need to show that any nonzero $\tilde{\mathfrak{a}}$-submodule N of V_k^{γ} contains $\overline{e^{\lambda}\otimes\xi_k^{\gamma}}$. Let $0 \neq x \in N \subset V_k^{\gamma}$ and write $x = \phi_k\left(\sum_{i=1}^{p} v_i\otimes\overline{w_i}\right)$ where $v_i \in V_{\dot{Q}}(\alpha)$ and $w_i \in W_k^{\gamma}(n\delta)$ for $1\le i\le p$. By (3.4.4) and the argument used in proposition 1.4.2(ii)⇒(iii) we can reduce to the case where $0 \neq y := \phi_k\left(c(e^{\alpha}\otimes 1)\otimes\overline{w_{i_0}}\right)\in N$, $1\le i_0\le p$ and $c\in\mathbb{C}^{\times}$. Finally, by lemma 3.4.3 and the fact that $W_k^{\gamma}(n\delta)$ is a cyclic irreducible $\mathcal{V}ir$-module with generator $\overline{e^{\lambda}\otimes\xi_k^{\gamma}}$ we are done. ∎

Corollary 3.4.5: For any $k\in\mathbb{Z}$ each of the irreducible components V_k^{γ}, $\gamma\in I_k$, of V_k is a highest weight $\tilde{\mathfrak{a}}$-module but is not a Verma module.

Proof: Fix $k\in\mathbb{Z}$ and $\gamma\in I_k$. It is clear that the vector $\overline{e^{\lambda}\otimes\xi_k^{\gamma}}$ is a highest weight vector in $V_k^{\gamma}(\lambda)$ with respect to $\tilde{\mathfrak{a}}$. Finally, that V_k^{γ} is not a Verma module follows from the fact that the abelian subalgebra $\mathfrak{a}(\mathbb{Z}\delta)$ belongs to the annihilator of the module V_k^{γ} (see (3.4.4)). ∎

VERTEX REPRESENTATIONS OF THE VIRASORO-HEISENBERG ALGEBRA

Introduction: In this chapter we finally return to the Generalized Heisenberg algebra $\mathcal{H} = \mathcal{H}(\mathring{Q},2)$ which was first introduced in chapter 1. In fact, we will be interested in a semi-direct product algebra $\tilde{\mathcal{H}} := \mathcal{V}ir \propto \mathcal{H}$.

In chapter 3 we focused on the subalgebra $\mathfrak{a} = \mathfrak{a}(Q)$ of $\mathcal{H}$ and we studied the structure of the Fock space $V_\Gamma(\lambda) = \mathbb{C}e^\lambda \otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma)_-)$ over $\tilde{\mathfrak{a}} = \mathcal{V}ir \propto \mathfrak{a}$. In order to extend this to a representation of the full $\tilde{\mathcal{H}}$, we will need to enlarge the representation space from $V_\Gamma(\lambda)$ to $H(\lambda) = \mathbb{C}[\lambda+\mathbb{Z}\delta] \otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma)_-)$ where $\mathbb{C}[\lambda+\mathbb{Z}\delta]$ is the group algebra of $\lambda+\mathbb{Z}\delta \cong \mathbb{Z}$. To represent the elements that are in $\mathcal{H}$ but not in $\mathfrak{a}$, we will need to introduce the vertex operators.

We will show that the $\tilde{\mathcal{H}}$-module $H(\lambda)$ is irreducible when $\lambda \in \Gamma\backslash Q$. If $\lambda \in Q$ $H(\lambda)$ is not irreducible and using the filtrations developed in chapters 2 and 3 we produce a corresponding filtration of $\tilde{\mathcal{H}}$-submodules $H^k(\lambda)$, $k \in \mathbb{Z}$. A slight twist occurs when we consider the subquotients $H_k(\lambda) := H^k(\lambda)/H^{k-1}(\lambda)$. In contrast to the subquotients $V_k(\lambda)$ over $\tilde{\mathfrak{a}}$, the $H_k(\lambda)$ are not completely reducible over $\tilde{\mathcal{H}}$. When $\lambda \in Q$, the operators in $\tilde{\mathcal{H}}$ cannot "separate" the elements in the enlarged group algebra $\mathbb{C}[\lambda+\mathbb{Z}\delta]$.

§ 4.1 Construction of the Virasoro-Heisenberg algebra $\tilde{\mathcal{H}} = \mathcal{V}ir \propto \mathcal{H}$

In this section $\dot{Q}$ denotes a (positive definite, even) lattice of type A_ℓ, D_ℓ or E_ℓ. The Generalized Heisenberg algebra $\mathcal{H} = \mathcal{H}(\dot{Q},2)$ introduced in ch.1 has linear basis $\{\gamma(m,n)\colon \gamma\in\dot{\mathfrak{h}},\ m,n\in\mathbb{Z}\} \cup \left\{z_{(1,0)}(m,n),\ z_{(0,1)}(m,0),\ z_{(1,0)}(0,0),\ m\in\mathbb{Z},\ n\in\mathbb{Z}\setminus\{0\}\right\}$ with multiplication

$$(4.1.1)\qquad \begin{cases} [\gamma(m,n),\gamma'(m',n')] = (\gamma|\gamma')\, z_{(m,n)}(m+m',n+n') \\ z_{(m,n)}(r,s) \quad \text{are central.} \end{cases}$$

Recall from (1.1.1) that the central elements $z_{(m,n)}(r,s)$ are linear in their subscript. We now consider a representation of $\mathcal{V}ir$ on $\mathcal{H}$ as follows. For $k\in\mathbb{Z}$ we let d_k be the unique derivation of $\mathcal{H}$ satisfying

$$(4.1.2)\qquad \begin{cases} d_k(\gamma(m,n)) := -m\,\gamma(m+k,n) \\ d_k(z_{(1,0)}(m,n)) := -(m+k)\, z_{(1,0)}(m+k,n),\ n\neq 0 \\ d_k(z_{(0,1)}(m,0)) := -m\ z_{(0,1)}(m+k,0) \\ d_k(z_{(1,0)}(0,0)) := 0 \end{cases}$$

We also define $z(\mathcal{H}) \equiv 0$. One now checks that $(d_k d_\ell - d_\ell d_k)(x) = [d_k,d_\ell](x)$ for any $x\in\mathcal{H}$. For example if $x = \gamma(m,n)$ then both sides become $m(\ell-k)\,\gamma(m+k+\ell,n)$. If $x = z_{(1,0)}(m,n)$ both sides become $(\ell-k)(k+\ell+m)\, z_{(1,0)}(k+\ell+m,n)$. Thus $\mathcal{H}$ is a module over $\mathcal{V}ir$ and we form the semi-direct product algebra $\tilde{\mathcal{H}} := \mathcal{V}ir \propto \mathcal{H}$ which we call the *Virasoro-Heisenberg algebra*.

Remark One should note that there is nothing mysterious about the definition of how $\mathcal{V}ir$ acts on $\mathcal{H}$ in (4.1.2). In a sense, these definitions are forced upon us by the known commutation relations of the vertex operators which we will introduce in the next section. The same comment could have been made about (3.1.2) in §3.1. Using proposition 1.1.1 (iii) and both (4.1.1) and (4.1.2) one can prove

Proposition 4.1.1: $\text{Centre}(\tilde{\mathcal{H}}) = \dot{\mathfrak{h}}(0) \oplus \mathbb{C}\, z_{(0,1)}(0,0) \oplus \mathbb{C}\, z_{(1,0)}(0,0) \oplus \mathbb{C}\mathfrak{z}$ and hence $\dim_{\mathbb{C}} \text{centre } \tilde{\mathcal{H}} = \ell+3$ where $\dim_{\mathbb{C}} \dot{\mathfrak{h}} = \ell = \text{rank } \dot{Q}$.

§4.2 Vertex Operator representations of $\tilde{\mathcal{H}}$

Let $\Gamma = \dot{Q} \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\mu$ be as in Ch.3 with $(\mu|\dot{Q}) = (\delta|\dot{Q}) = (\delta|\delta) = (\mu|\mu) = 0$ and $(\delta|\mu) = 1$. Let $\varepsilon : Q \times Q \longrightarrow \langle \pm 1 \rangle$ be a bimultiplicative map satisfying

$$(4.2.1) \qquad \begin{cases} \text{CC (i)} & \varepsilon(\alpha,\alpha) := (-1)^{(\alpha|\alpha)/2} \\ \text{CC (ii)} & \varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha) = (-1)^{(\alpha|\beta)} \\ \text{CC (iii)} & \varepsilon(\alpha,\delta) = 1 \end{cases}$$

where $\alpha,\beta \in Q$. Extend ε to a bimultiplicative map $\varepsilon : Q \times \Gamma \longrightarrow \langle \pm 1 \rangle$ The existence of cocycles (or equivalently, central extensions) of even geometic lattices is discussed in appendix C.

For $\gamma \in \Gamma$ let e^{γ} be a symbol and form the vector space $\mathbb{C}[\Gamma]$ with $\mathbb{C}$-basis $\{e^{\gamma} : \gamma \in \Gamma\}$. Then $\mathbb{C}[\Gamma]$ contains the subspace $\mathbb{C}[Q] := \coprod_{\gamma \in Q} \mathbb{C}e^{\gamma}$. We give $\mathbb{C}[Q]$ a twisted group algebra structure by defining $e^{\alpha}e^{\beta} := \varepsilon(\alpha,\beta)\, e^{\alpha+\beta}$, $\alpha,\beta \in Q$ - see [MP-2] chapter 3. Then $\mathbb{C}[\Gamma]$ becomes a $\mathbb{C}[Q]$-module by defining $e^{\alpha}e^{\gamma} := \varepsilon(\alpha,\gamma)\, e^{\alpha+\gamma}$, $\alpha \in Q$, $\gamma \in \Gamma$. Now form the *Full Fock space*

$$(4.2.2) \qquad V(\Gamma) := \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma)_-)$$

associated to Γ.

Note that as $\mathbb{C}$-spaces, $V(\Gamma) = \coprod_{\lambda \in \Gamma} V_{\Gamma}(\lambda)$ where $V_{\Gamma}(\lambda) = \mathbb{C}e^{\lambda} \otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma)_-)$ is the $\tilde{\mathfrak{a}}$-module that we studied in chapter 3. We now use this decomposition of $V(\Gamma)$ into $\tilde{\mathfrak{a}}$-modules to make $V(\Gamma)$ into an $\tilde{\mathfrak{a}}$-module. To extend this action of $\tilde{\mathfrak{a}}$ on $V(\Gamma)$ to an action of the full algebra $\tilde{\mathcal{H}}$ on $V(\Gamma)$ we need to introduce the vertex Operators.

Let z be a complex variable and $\alpha \in Q$. Define

$$T_{\pm}(\alpha,z) := -\sum_{n \gtrless 0} \frac{1}{n}\alpha(n)z^{-n}. \tag{4.2.3}$$

Then the *Vertex Operator*, $X(\alpha,z)$, for α on $V(\Gamma)$ is defined by

$$\begin{cases} X(\alpha,z) := z^{(\alpha|\alpha)/2} \exp T(\alpha,z) \quad \text{where} \\ \exp T(\alpha,z) := \exp T_-(\alpha,z)\, e^{\alpha} z^{\alpha(0)} \exp T_+(\alpha,z) \quad \text{and} \\ z^{\alpha(0)}(e^{\lambda}\otimes f) := z^{(\alpha|\lambda)}(e^{\lambda}\otimes f),\ f \in S(a(\Gamma)_-). \end{cases} \tag{4.2.4}$$

In [MP-1] one finds a careful analysis of these operators and the authors show that for any $z \in \mathbb{C}^{\times}$, $X(\alpha,z)$ maps $V(\Gamma)$ into $\mathbb{C}[\Gamma]\otimes_{\mathbb{C}} \overline{S(a(\Gamma)_-)}$ where $\overline{S(a(\Gamma)_-)}$ denotes the completion to formal power series in $S(a(\Gamma)_-)$. Moreover, $X(\alpha,z)$ can be formally expanded in powers of z to give

$$X(\alpha,z) = \sum_{n \in \mathbb{Z}} X_n(\alpha) z^{-n}$$

and the coefficients $X_n(\alpha)$ are called *moments*. In the next section we will carry out some explicit calculations with the $X_n(\alpha)$ and we will see that they are operators on $V(\Gamma)$. Moreover, we will see that for any $f \in S(a(\Gamma)_-)$, one has $X_n(\alpha)\ (e^{\lambda}\otimes f) = e^{\lambda+\alpha}\otimes f'$ where $f' \in S(a(\Gamma)_-)$. Thus, in the decomposition of the full Fock space $V(\Gamma) = \coprod_{\lambda \in \Gamma} V_{\Gamma}(\lambda)$ one can view the moments $X_n(\alpha)$ as operators which move an element in the "λ-stalk" to an element in the "$(\lambda+\alpha)$-stalk".

In order to determine the commutation relations of the moments (which is, for us, the key to the construction of the vertex operators), one can employ the techniques of residue calculus and interpret the operator $X_n(\alpha)$ by the

complex integral $\int_{\gamma} X(\alpha,z)\ \frac{z^n}{z}\,dz$ where γ is a closed curve around the origin. One finds the following identities:

CR.0 $[\alpha(k),X_n(\beta)] = (\alpha|\beta)\ X_{n+k}(\beta)$

CR.1 $[X_m(\alpha),X_n(\beta)] = 0\ ,\quad (\alpha|\beta)\geq 0$

CR.2 $[X_m(\alpha),X_n(\beta)] = \varepsilon(\alpha,\beta)\ X_{n+m}(\alpha+\beta)\ ,\quad (\alpha|\beta)=-1$

CR.3 If $(\alpha|\alpha) = (\beta|\beta) = -(\alpha|\beta) = 2$ then

$$[X_m(\alpha),X_n(\beta)] = \varepsilon(\alpha,\beta)\ \{mX_{n+m}(\alpha+\beta) + \sum_{k\in\mathbb{Z}} :\alpha(k)\ X_{m+n-k}(\alpha+\beta):\}$$

where $:\alpha(k)X_{m+n-k}(\beta): := \begin{cases} \alpha(k)X_{m+n-k}(\beta) & \text{if } k \leq m+n-k \\ X_{m+n-k}(\beta)\alpha(k) & \text{if } k > m+n-k \end{cases}$

CR.4 $[L_k,X(\alpha,z)] = z^k\left\{\frac{k}{2}\ (\alpha|\alpha) + z\frac{d}{dz}\right\} X(\alpha,z).$

Proof: See [MP-1] for **CR.0 - CR.3** and [GO] or [KF] for **CR.4**. ∎

Next we will state the main result from [MEY] which gives vertex representations of the toroidal Lie algebra $\tau_{[2]}$ defined in ch.1 §2. In $\tau_{[2]}$ we can identify an affine Kac-Moody algebra $\dot{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[s,s^{-1}] \oplus \mathbb{C}c$. Its root system is denoted Δ, its root lattice Q and its set of real roots Δ^{re}.

Theorem : Let $\mathfrak{t}$ be the Lie algebra of operators on $V(\Gamma)$ generated by the moments $X_m(\alpha)$, $\alpha\in\Delta^{re}\subset Q$ where $m\in\mathbb{Z}$ and Δ^{re} denotes the set of real roots in Δ. Then $\mathfrak{t}$ is isomorphic to $\tau_{[2]}$ under the following assignment:

$$
(4.2.5)\qquad \begin{cases} e_i \otimes s^m t^n \longmapsto X_m(\alpha_i + n\delta) \\ -f_i \otimes s^m t^n \longmapsto X_m(-\alpha_i + n\delta) \\ z_{(1,0)}(m,n) \longmapsto X_m(n\delta),\ n \neq 0 \\ z_{(0,1)}(m,0) \longmapsto \delta(m) \\ z_{(1,0)}(0,0) \longmapsto I \end{cases}
$$

Definition: For $\gamma \in \mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} \dot{Q}$ and $m,n \in \mathbb{Z}$ define

$$
(4.2.6)\qquad T_m^{\gamma}(n\delta) := \sum_{k \in \mathbb{Z}} :\gamma(k)\, X_{-k+m}(n\delta):
$$

where the normal ordering is defined as in **CR.3**.

Although the right side of $T_m^{\gamma}(n\delta)$ is an infinite sum, for any fixed $v \in V(\Gamma)$, only finitely many terms can act nontrivially. Indeed, when $k \ll 0$, $-k+m \gg 0$ and then $X_{-k+m}(n\delta)$ kills v (see 3.4.3). If $k \gg 0$ then $\gamma(k)$ kills v. Thus $T_m^{\gamma}(n\delta)$ makes sense as an operator on $V(\Gamma)$. Note that when $n=0$, $T_m^{\gamma}(0)$ acts as a nonzero multiple of the operator $\gamma(m)$. Finally, it is useful to observe that $T_m^{\gamma}(n\delta)$ is linear in its superscript:

$$
(4.2.7)\qquad \begin{cases} T_m^{\gamma}(n\delta) + T_m^{\gamma'}(n\delta) = T_m^{\gamma+\gamma'}(n\delta), & \gamma,\gamma' \in \mathfrak{h} \\ cT_m^{\gamma}(n\delta) = T_m^{c\gamma}(n\delta), & c \in \mathbb{C}. \end{cases}
$$

Proposition 4.2.1 $V(\Gamma)$ is an $\mathcal{H}$-module under the following correspondences:

$$(4.2.8)\qquad \begin{cases} \gamma(m,n) \longmapsto T_m^{\gamma}(n\delta) \\ z_{(1,0)}{}^{(m,n)} \longmapsto X_m(n\delta),\ n\neq 0 \\ z_{(0,1)}{}^{(m,0)} \longmapsto \delta(m) \\ z_{(1,0)}{}^{(0,0)} \longmapsto I \end{cases}$$

Proof: By the theorem, it suffices to check only the first of these. Indeed, by the definition of multiplication in $\tau_{[2]}$ (see §1.2) and by **CR.3** we have

$$\begin{aligned} \phi(h_i\otimes s^m t^n) &= \phi\left([e_i\otimes 1, f_i\otimes s^m t^n]\right) \\ &= -[X_0(\alpha_i+0\delta), X_m(-\alpha_i+n\delta)] \\ &= -\varepsilon(\alpha_i,-\alpha_i)\sum_{k\in\mathbb{Z}} :\alpha_i(k)\ X_{-k+m}(n\delta): \\ &= T_m^{\alpha_i}(n\delta). \end{aligned}$$

Now let $\gamma\in\dot{\mathfrak{h}}$ be written $\gamma = \sum_{i=1}^{\ell} c_i h_i$, $c_i\in\mathbb{C}$. Then by proposition 1.2.1, $\gamma(m,n)$ corresponds to $\gamma\otimes s^m t^n \in \tau_{[2]}$ which can be written $\sum_{i=1}^{\ell} c_i(h_i\otimes s^m t^n)$. Now, by the above calculation, the result is clear. ∎

Recall from §2 that $V(\Gamma)$ can be decomposed into the $\tilde{\mathfrak{a}}$-modules $V_\Gamma(\lambda)$. Our next goal is to represent $\tilde{\mathcal{H}}$ on $V(\Gamma)$. We extend the representation of $\mathcal{H}$ on $V(\Gamma)$ to a representation of $\tilde{\mathcal{H}}$ on $V(\Gamma)$ by letting the Virasoro generator d_k act on each $V_\Gamma(\lambda)$ by the operator L_k from chapter 3 and extend the action linearly. Then on each $V_\Gamma(\lambda)$, $\mathfrak{z}$ acts as $\ell+2$ and we have

Proposition 4.2.2 $V(\Gamma)$ is an $\tilde{\mathcal{H}}$-module.

Proof: By proposition 2.2.1, for any $a\in Q$, $[L_k,a(m)] = -ma(m+k)$. In particular $[L_k,\delta(m)] = -m\delta(m+k)$ and hence by (4.2.8) we have $d_k\left(z_{(0,1)}(m,0)\right) = -m\ z_{(0,1)}(m+k,0)$, as required by (4.1.2). Similarly, by **CR.4** we have $[L_k,X_m(n\delta)] = -(m+k)\ X_{m+k}(n\delta)$ and hence by (4.2.8) this means $d_k\left(z_{(1,0)}(m,n)\right) = -(m+k)z_{(1,0)}(m+k,n)$ as required by (4.1.2). Finally,

$$[L_k,T_m^{\gamma}(n\delta)] = \sum_{\ell\in\mathbb{Z}} [L_k,\gamma(\ell)\ X_{-\ell+m}(n\delta)]$$

$$= \sum_{\ell\in\mathbb{Z}}\left\{[L_k,\gamma(\ell)]X_{-\ell+m}(n\delta) + \gamma(\ell)[L_k,X_{-\ell+m}(n\delta)]\right\}$$

$$= \sum_{\ell\in\mathbb{Z}}\left\{-\ell\ \gamma(k+\ell)\ X_{-\ell+m}(n\delta) + (\ell-m-k)\ \gamma(\ell)\ X_{k+m-\ell}(n\delta)\right\}$$

$$= \sum_{\ell\in\mathbb{Z}}\left\{-\ell\ \gamma(k+\ell)\ X_{-\ell+m}(n\delta) + (\ell-m)\ \gamma(\ell+k)\ X_{-\ell+m}(n\delta)\right\}$$

$$= -m\sum_{\ell\in\mathbb{Z}}\gamma(k+\ell)\ X_{-\ell+m}(n\delta) = -m\ T_{m+k}^{\gamma}(n\delta). \qquad \blacksquare$$

§ 4.3 The $\tilde{\mathcal{H}}$-module $H(\lambda)$

For $\lambda\in\Gamma$ let $H(\lambda) := \mathbb{C}[\lambda+\mathbb{Z}\delta]\otimes_{\mathbb{C}}S(\mathfrak{a}(\Gamma)_-)$ where $\mathbb{C}[\lambda+\mathbb{Z}\delta]$ is the twisted group algebra of the subgroup $\lambda+\mathbb{Z}\delta$ of Γ. Then $\mathbb{C}[\lambda+\mathbb{Z}\delta]$ has the $\mathbb{C}$-basis $\{e^{\lambda+n\delta}:n\in\mathbb{Z}\}$. We have the decomposition

$$V(\Gamma) = \coprod_{\lambda} H(\lambda)$$

where λ ranges through a complete set of representatives of $\Gamma/\mathbb{Z}\delta$.

The goal of this section is to show that for any $\lambda\in\Gamma$, $H(\lambda)$ is an $\tilde{\mathcal{H}}$-submodule of $V(\Gamma)$. Since $T_m^{\alpha}(n\delta) = \sum_{k\in\mathbb{Z}} :\alpha(k)X_{-k+m}(n\delta):$ where $\alpha\in\dot{Q}$ it suffices to show that for any $\lambda\in\Gamma$, $p\in\mathbb{Z}$, $\gamma_1,\dots,\gamma_s \in \Gamma$ and $q_1,\dots,q_s >0$ we have

$$X_m(n\delta)\cdot\left(e^{\lambda+p\delta}\otimes\gamma(-q_1)\cdots\gamma(-q_s)\right) \in H(\lambda). \tag{4.3.1}$$

In fact, this is not hard to see but the explicit expression for (4.3.1) will soon be useful to us so we calculate it now.

In most of our calculations we will be working with a fixed given $\lambda\in\Gamma$ and since $\varepsilon(\lambda,\delta) \in \{\pm1\}$ we will assume without loss in generality that $\varepsilon(\lambda,\delta) = 1$ unless otherwise stated. Of course, in any expression with λ varying, one would need to carry the cocycles. Recall $T_{\pm}(\alpha,z) = -\sum_{r\gtrless 0} \frac{1}{r}\alpha(r)\,z^{-r}$. For $\alpha\in\Gamma$ define the *elementary Schur polynomials* $S_r(\alpha)$, $r\in\mathbb{Z}$ by the expressions

$$\begin{cases} \exp T_-(\alpha,z) =: \sum_{r=0}^{\infty} S_r(\alpha)\,z^r \\ S_r(\alpha) := 0\,, \quad r<0 \end{cases} \tag{4.3.2}$$

Then $\sum_{m\in\mathbb{Z}} X_m(n\delta)\, z^{-m}\cdot(e^{\lambda+p\delta}\otimes 1) = X(n\delta,z)\cdot(e^{\lambda+p\delta}\otimes 1)$

$$= \exp T_-(n\delta,z)\; e^{n\delta}\; z^{n\delta(0)}(e^{\lambda+p\delta}\otimes 1)$$

$$= \sum_{r=0}^{\infty}\left(e^{\lambda+(n+p)\delta}\otimes S_r(n\delta)\right)z^{r+n(\lambda|\delta)}$$

Thus letting $N := (\lambda|\delta)$ and matching the powers of z by putting $-m = r + n(\lambda|\delta)$ we have

$$X_m(n\delta)\cdot(e^{\lambda+p\delta}\otimes 1) = e^{\lambda+(p+n)\delta}\otimes S_{-(m+nN)}(n\delta) \tag{4.3.3}$$

which establishes (4.3.1) in the case s=0. Next,

$$\sum_{m\in\mathbb{Z}} X_m(n\delta)\cdot\left(e^{\lambda+p\delta}\otimes \gamma_1(-q_1)\cdots\gamma_s(-q_s)\right)z^{-m}$$

$$= X(n\delta,z)\cdot\left(e^{\lambda+p\delta}\otimes \gamma_1(-q_1)\cdots\gamma_s(-q_s)\right)$$

$$= \exp T_-(n\delta,z)\; e^{n\delta}z^{n\delta(0)}\exp T_+(n\delta,z)\cdot\left(e^{\lambda+p\delta}\otimes \gamma_1(-q_1)\cdots\gamma_s(-q_s)\right).$$

But $T_+(n\delta,z)\cdot\left(e^{\lambda+p\delta}\otimes \gamma_1(-q_1)\cdots\gamma_s(-q_s)\right)$

$$= -\sum_{r>0}\frac{1}{r}\,n\delta(r)z^{-r}\cdot\left(e^{\lambda+p\delta}\otimes \gamma_1(-q_1)\cdots\gamma_s(-q_s)\right)$$

$$= -n\sum_{i=1}^{s}(\delta|\gamma_i)\left(e^{\lambda+p\delta}\otimes \gamma(-q_1)\cdots\overline{\gamma_i(-q_i)}\cdots\gamma_s(-q_s)\right)z^{-q_i}$$

where the overbar denotes omission.

Similarly, $T_+^2(n\delta,z)\cdot\left(e^{\lambda+p\delta}\otimes \gamma_1(-q_1)\cdots\gamma_s(-q_s)\right)$

$$= +2n^2\sum_{i=1}^{s}\sum_{j\neq i}(\delta|\gamma_i)(\delta|\gamma_j)\left(e^{\lambda+p\delta}\otimes \gamma(-q_1)\cdots\overline{\gamma_i(-q_i)}\cdots\overline{\gamma_j(-q_j)}\cdots\gamma_s(-q_s)\right)z^{-(q_i+q_j)}$$

$$\vdots$$

$$T_+^{s-1}(n\delta,z)\cdot\left(e^{\lambda+p\delta}\otimes \gamma_1(-q_1)\cdots\gamma_s(-q_s)\right)$$

$$= (-1)^{s-1}(s-1)!\ n^{s-1}\sum_{i=1}^{s}\left(\prod_{j\neq i}(\delta|\gamma_j)\right)\left(e^{\lambda+p\delta}\otimes\gamma_i(-q_i)\right)\ z^{-\left(\sum_{j\neq i}q_j\right)}\quad\text{and finally}$$

$$T_+^{s}(n\delta,z)\cdot\left(e^{\lambda+p\delta}\otimes\gamma_1(-q_1)\cdots\gamma_s(-q_s)\right)$$

$$= (-1)^{s}s!\ n^{s}\left(\prod_{i=1}^{s}(\delta|\gamma_i)\right)\ (e^{\lambda+p\delta}\otimes 1)\ z^{-\left(\sum_{i=1}^{s}q_i\right)}\ .$$

Thus $T_+^{s+c}(n\delta,z)\cdot\left(e^{\lambda+p\delta}\otimes \gamma_1(-q_s)\cdots\gamma_s(-q_s)\right) = 0$, $c>0$. Now arguing as before

to extract $X_m(n\delta)\cdot\left(e^{\lambda+p\delta}\otimes \gamma_1(-q_1)\cdots\gamma(-q_s)\right)$ we have

$$X_m(n\delta)\cdot\left(e^{\lambda+p\delta}\otimes \gamma_1(-q_1)\cdots\gamma_s(-q_s)\right)$$

$$= \left(e^{\lambda+(p+n)\delta}\otimes S_{-(m+nN)}(n\delta)\ \gamma_1(-q_1)\cdots\gamma_s(-q_s)\right)$$

$$- \left\{n\sum_{i=1}^{s}(\delta|\gamma_i)\left(e^{\lambda+(p+n)\delta}\otimes S_{q_i-(m+nN)}(n\delta)\ \gamma_1(-q_1)\cdots\overline{\gamma_i(-q_i)}\cdots\gamma_s(-q_s)\right)\right.$$

$$+2n^2\sum_{i=1}^{s}\sum_{j\neq i}(\delta|\gamma_i)(\delta|\gamma_j)\left(e^{\lambda+(p+n)\delta}\otimes S_{q_{ij}}(n\delta)\gamma_1(-q_1)\cdots\overline{\gamma_i(-q_i)}\cdots\overline{\gamma_j(-q_j)}\cdots\gamma_s(-q_s)\right)$$

(where $q_{ij} := (q_i+q_j)-(m+nN)$)

$$+\cdots+ (s-1)!\ n^{(s-1)}\sum_{i=1}^{s}\left(\prod_{j\neq i}(\delta|\gamma_j)\right)\left(e^{\lambda+(p+n)\delta}\otimes S_{\left(\sum_{j\neq i}q_j\right)-(m+nN)}(n\delta)\ \gamma_i(-q_i)\right)$$

$$+ s!\ n^s\left(\prod_{i=1}^{s}(\delta|\gamma_i)\right)\left(e^{\lambda+(p+n)\delta}\otimes S_{\left(\sum_{i=1}^{s}q_i\right)-(m+nN)}(n\delta)\right\} \qquad (4.3.4)$$

Now in particular, for $q>0$ and $\gamma\in\Gamma$ we have

$$X_m(n\delta)\cdot(e^{\lambda+p\delta}\otimes\gamma(-q)) = e^{\lambda+(p+n)\delta}\otimes S_{-(m+nN)}(n\delta)$$

$$- n(\delta|\gamma)\left(e^{\lambda+(p+n)\delta}\otimes S_{q-(m+nN)}(n\delta)\right) \in H(\lambda) \qquad (4.3.5).$$

Proposition 4.3.1 For any $\lambda\in\Gamma$, $H(\lambda)$ is an $\tilde{\mathcal{H}}$-submodule of $V(\Gamma)$.

Proof: This is clear from the above calculations and the discussion prior to them. ∎

§4.4 The Structure of $H(\lambda)$, $\lambda\in\Gamma\backslash Q$

Combining the calculations in the previous section with the main result of §3.3 we are able to prove the following result.

Proposition 4.4.1 For $\lambda\in\Gamma\backslash Q$, $H(\lambda)$ is an irreducible $\tilde{\mathcal{H}}$-module.

Proof: Our method of proof will be to show (i) $e^{\lambda}\otimes 1$ generates $H(\lambda)$ and (ii) every nonzero submodule of $H(\lambda)$ contains $e^{\lambda}\otimes 1$. First note that $H(\lambda) = \coprod_{n\in\mathbb{Z}} \mathbb{C}e^{\lambda+n\delta}\otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma)_-) = \coprod_{n\in\mathbb{Z}} V(\lambda+n\delta)$. Now since $\lambda\in\Gamma\backslash Q$ it is clear that for each $n\in\mathbb{Z}$, $\lambda+n\delta \in \Gamma\backslash Q$. Thus by proposition 3.3.4 $\mathbb{C}e^{\lambda+n\delta}\otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma)_-)$ is an irreducible $\tilde{\mathfrak{a}}$-module generated by $e^{\lambda+n\delta}\otimes 1$. To show (i) it suffices to check that for each $n\in\mathbb{Z}$, $e^{\lambda+n\delta}\otimes 1$ is in the submodule generated by $e^{\lambda}\otimes 1$. Indeed given $n\in\mathbb{Z}$ choose $m\in\mathbb{Z}$ so that $m+nN = 0$. Then by (4.3.3) we have $X_m(n\delta)\cdot(e^{\lambda}\otimes 1) = e^{\lambda+n\delta}\otimes 1$.

Next let R be a nonzero submodule of $H(\lambda)$ and let $0 \neq z := \sum_{i=1}^{s} e^{\lambda+k_i\delta}\otimes f_i \in R$, $k_i\in\mathbb{Z}$, $f_i \in S(\mathfrak{a}(\Gamma)_-)$ and $s\geq 1$. Without loss of generality we may assume that the k_i are distinct. We begin by "differentiating out" all indeterminates of the form $\mu(-n),n>0$ using $\delta(n)$ and those of the form $\alpha(-m)$, $\alpha\in\dot{Q}$, $m>0$ using $\alpha(m)$. Thus we may assume $f_i \in D = S\left(\coprod_{m>0}\mathbb{C}\delta(-m)\right)$, $1\leq i\leq r$. Now arguing as in the proof of lemma 3.3.2 we can reduce z to a nonzero element $x = \sum_{i=1}^{r} c_i e^{\lambda+k_i\delta}\otimes 1 \in R$ where $c_i\in\mathbb{C}^{\times}$, $r\leq s$. We say that x has *length* r since it has r distinct k_i's. If $r=1$ then $x = c_1 e^{\lambda+k_1\delta}\otimes 1$ and by (4.3.3) $X_{k_1N}(-k_1\delta)\cdot(c_1 e^{\lambda+k_1\delta}\otimes 1) = c_1(e^{\lambda}\otimes 1)$ as required. If $r\geq 1$ then by induction it suffices to show that we can shorten the length of

x by exactly one.

Let $m := (\lambda|\lambda)/2 \in \mathbb{Z}$ and choose $k \in \mathbb{Z}$ so that $n_1 := m + (k+k_1)N < 0$. Write $n_1 =: -n$, $n>0$ and consider $y := L_0 \cdot \delta(-n) \cdot X_{kN}(-k\delta) \cdot x$. We claim that y has length r-1. Write $x = c_1 e^{\lambda+k_1\delta} \otimes 1 + x'$ where $x' := \sum_{i=2}^{r} c_i e^{\lambda+k_i\delta} \otimes 1$. Then $y = c_1 L_0 \cdot \delta(-n) \cdot X_{kN}(-k\delta) \cdot (e^{\lambda+k_1\delta} \otimes 1) + L_0 \cdot \delta(-n) \cdot X_{kN}(-k\delta) \cdot x'$. It suffices to show that the first term is zero and no other term is. Indeed, for $1 \le i \le r$ and $\varepsilon_i := \varepsilon(-k\delta, \lambda+k_i\delta)$

$$L_0 \cdot \delta(-n) \cdot X_{kN}(-k\delta) \cdot (e^{\lambda+k_i\delta} \otimes 1)$$

$$= \varepsilon_i \, L_0 \cdot \left(e^{\lambda+(k_i+k)\delta} \otimes \delta(-n)\right)$$

$$= \varepsilon_i \left(\frac{(\lambda|\lambda)}{2} + (k_i+k)(\lambda|\delta) + n \right) \left(e^{\lambda+(k_i+k)\delta} \otimes \delta(-n)\right)$$

$= \varepsilon_i \, (m + (k_i+k)N + n) \left(e^{\lambda+(k_i+k)\delta} \otimes \delta(-n)\right)$. Now the coefficient $m+(k+k_i)+n = 0 \iff k=k_i$ by the choice of n. ∎

§4.5 The Structure of $H(\lambda)$, $\lambda\in Q$

In this section we present the analogue of the material discussed in §3.4. For $k\in\mathbb{Z}$ we introduce the following $\mathbb{C}$-subspaces of $H(\lambda)$:

$$H^k(\lambda) := \mathbb{C}[\lambda+\mathbb{Z}\delta]\otimes_{\mathbb{C}} S(\mathfrak{a}(\dot{Q})_-)\Theta^k = \coprod_{n\in\mathbb{Z}} V^k(\lambda+n\delta).$$

Proposition 4.5.1 For and $k\in\mathbb{Z}$, $H^k(\lambda)$ is an $\tilde{\mathcal{H}}$-submodule of $H(\lambda)$.

Proof: Fix $k\in\mathbb{Z}$. By proposition 3.4.2, it suffices to check that $H^k(\lambda)$ is invariant under $X_m(n\delta)$, $m\in\mathbb{Z}$, $n\in\mathbb{Z}\setminus\{0\}$. Since $\Theta^k = \coprod_{j\geq 0}\left(\Omega_{\leq(j+k)}\Delta_j\right)$ this follows immediately from (4.3.4). ∎

Next let $H_k(\lambda) := H^k(\lambda)/H^{k-1}(\lambda)$. Then, as $\mathbb{C}$-spaces, $H_k(\lambda) \overset{\Phi}{\cong} \coprod_{n\in\mathbb{Z}} V_k(\lambda+n\delta)$ where $\Phi\left(\overline{\sum_n e^{\lambda+n\delta}\otimes f_n}\right) := \sum_n\left(\overline{e^{\lambda+n\delta}\otimes f_n}\right)$ and at most finitely many f_n are nonzero. By transfering the $\tilde{\mathcal{H}}$-module structure from $H_k(\lambda)$ to $\coprod_{n\in\mathbb{Z}} V_k(\lambda+n\delta)$ we find from (4.3.4) that

$$X_q(p\delta)\cdot\left(\overline{e^{\lambda+n\delta}\otimes f}\right) = \begin{cases} 0\ , & q\neq 0 \\ \varepsilon\ \overline{e^{\lambda+(n+p)}\otimes f}, & q=0 \end{cases} \tag{4.5.2}$$

where $\varepsilon := \varepsilon(p\delta,\lambda+n\delta)$ and $f\in S(\mathfrak{a}(\dot{Q})_-)\Theta^k$ and $X_q(p\delta)$ acts linearly across the direct sum $\coprod_{n\in\mathbb{Z}} V_k(\lambda+n\delta)$. In the sequel we will identify $H_k(\lambda)$ and $\coprod_{n\in\mathbb{Z}} V_k(\lambda+n\delta)$ as $\tilde{\mathcal{H}}$-modules.

Fix $k\in\mathbb{Z}$ and for $\gamma\in I_k$ define

$$H_k^\gamma(\lambda) := \mathfrak{U}(\tilde{\mathcal{H}})\cdot\left(\overline{e^\lambda\otimes\xi_k^\gamma}\right) \tag{4.5.3}$$

Proposition 4.5.2 For any $k\in\mathbb{Z}$ and $\gamma\in I_k$, $H_k^{\gamma}(\lambda) = \coprod_{n\in\mathbb{Z}} V_k^{\gamma}(\lambda+n\delta)$ and consequently, $H_k(\lambda) = \coprod_{\gamma\in I_k} H_k^{\gamma}(\lambda)$.

Proof: The result is clear from (4.5.2) and proposition 3.4.4.. ∎

§ 4.6 Verma-type Modules for $\tilde{\mathcal{H}}$

For each $\tilde{N}\in\mathbb{Z}$ we will first introduce subalgebras $\tilde{\mathcal{H}}_-$, $\tilde{\mathcal{H}}_0$ and $\tilde{\mathcal{H}}_+$ of $\tilde{\mathcal{H}}$ which together decompose $\tilde{\mathcal{H}}$ but do not give a triangular decomposition in the sense of [MP-2]. From this decomposition we construct an $\tilde{\mathcal{H}}$-module which is reminiscent of the construction of a Verma module for a Lie algebra with triangular decomposition.

Letting $\mathcal{B}$ denote the subalgebra $\left(\tilde{\mathcal{H}}_0 \oplus \tilde{\mathcal{H}}_+\right)$ of $\tilde{\mathcal{H}}$, we consider elements $\tilde{\lambda} \in \tilde{\mathcal{H}}_0^{\,*}$ where * does not mean the usual $\mathbb{C}$-dual of linear functionals but $\tilde{\lambda} \in \tilde{\mathcal{H}}_0^{\,*}$ means that $\tilde{\lambda}$ is a linear map $\tilde{\lambda} : \tilde{\mathcal{H}}_0 \longrightarrow \mathbb{C}[\mathbb{Z}\delta]$ where $\mathbb{C}[\mathbb{Z}\delta]$ is the group algebra of the infinite cyclic group $\mathbb{Z}\delta$. Using $\tilde{\lambda}$ we make $\mathbb{C}[\mathbb{Z}\delta]$ into a $\mathcal{B}$-module and form the induced $\tilde{\mathcal{H}}$-module $M(\tilde{\lambda},\tilde{N}) := \mathcal{U}(\tilde{\mathcal{H}})\otimes_{\mathcal{U}(\mathcal{B})}\mathbb{C}[\mathbb{Z}\delta]$.

The goal of this section is to identify the irreducible $\tilde{\mathcal{H}}$-module $H(\lambda)$, $\lambda\in\Gamma\backslash Q$ from §4.4 as a quotient of some $M(\tilde{\lambda},\tilde{N})$. As in earlier sections of the chapter, in our calculations we will often omit the cocycle $\varepsilon(\alpha,\lambda)$ when λ is fixed.

Fix $\tilde{N}\in\mathbb{Z}$ and consider the following subspaces of $\mathcal{H}$.

$\mathcal{H}_\pm := \langle \gamma(m,n),\ z_{(1,0)}(m,n),\ z_{(0,1)}(k,0) : m+n\tilde{N} \gtrless 0,\ k\gtrless 0 \rangle$

$\mathcal{H}_0 := \langle \gamma(m,n),\ z_{(1,0)}(m,n),\ z_{(0,1)}(0,0),\ z_{(1,0)}(0,0) : m+n\tilde{N} = 0 \rangle$

where the angle brackets $\langle\ \rangle$ denote the linear $\mathbb{C}$-span of the elements enclosed.

Now define $\tilde{\mathcal{H}}_\pm := \mathcal{H}_\pm \oplus \left(\coprod_{n\gtrless 0} \mathbb{C}d_n\right)$ and $\tilde{\mathcal{H}}_0 := \mathcal{H}_0 \oplus \mathbb{C}d_0 \oplus \mathbb{C}\mathfrak{z}$

Proposition 4.6.1 (i) As $\mathbb{C}$-spaces, $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_- \oplus \tilde{\mathcal{H}}_0 \oplus \tilde{\mathcal{H}}_+$.

(ii) $\tilde{\mathcal{H}}_\pm$ and $\tilde{\mathcal{H}}_0$ are subalgebras of $\tilde{\mathcal{H}}$.

Proof: (i) follows from Proposition 1.3.1. For (ii) consider for example $\gamma(m,n)$, $\gamma'(m',n')$, $z_{(1,0)}(m,n)$ and $d_k \in \tilde{\mathcal{H}}_+$. Then we have

(a) $[\gamma(m,n),\gamma'(m',n')] = (\gamma|\gamma')(m\, z_{(1,0)}(m+m',n+n') + n\, z_{(0,1)}(m+m',n+n'))$.

(b) $[d_k,\gamma(m,n)] = -m\,\gamma(m+k,n)$ and

(c) $[d_k,z_{(1,0)}(m,n)] -(m+k)\, z_{(1,0)}(m+k,n)$.

Since $m+n\tilde{N} > 0$, $m'+n'\tilde{N} > 0$ and $k>0$ it follows that $(m+m') + (n+n')\tilde{N} > 0$ and $(m+k)+n\tilde{N} = k+(m+n\tilde{N}) > 0$. Thus each of (a), (b) and (c) belong to $\tilde{\mathcal{H}}_+$. Similarly for $\tilde{\mathcal{H}}_-$ and $\tilde{\mathcal{H}}_0$. This proves the proposition. ∎

Next consider the group algebra $\mathbb{C}[\mathbb{Z}\delta]$ of the infinite cyclic group $\mathbb{Z}\delta$ and let $\tilde{\lambda} : \tilde{\mathcal{H}}_0 \longrightarrow \mathbb{C}[\mathbb{Z}\delta]$ be a $\mathbb{C}$-linear map. Define $\mathcal{B} := \tilde{\mathcal{H}}_0 \oplus \tilde{\mathcal{H}}_+$ and make $\mathbb{C}[\mathbb{Z}\delta]$ into a $\mathcal{B}$-module by setting

$$(4.6.1) \qquad \begin{cases} \tilde{\mathcal{H}}_+ \cdot e^{p\delta} \equiv 0 \\ h \cdot e^{p\delta} := \tilde{\lambda}(h)\, e^{p\delta},\ p\in\mathbb{Z},\ h\in\tilde{\mathcal{H}}_0. \end{cases}$$

Now set $M(\tilde{\lambda},\tilde{N}) := \mathcal{U}(\tilde{\mathcal{H}})\otimes_{\mathcal{U}(\mathcal{B})}\mathbb{C}[\mathbb{Z}\delta]$. Our goal here is to identify the irreducible $\tilde{\mathcal{H}}$-module $H(\lambda)$, $\lambda\in\Gamma\backslash Q$ from §4.4 as a quotient of some $M(\tilde{\lambda},\tilde{N})$. Let $\lambda\in\Gamma$ be arbitrary and set $\tilde{N} := (\lambda|\delta)$. Then define $\tilde{\lambda} : \tilde{\mathcal{H}}_0 \longrightarrow \mathbb{C}[\mathbb{Z}\delta]$ by

(4.6.2)
$$\begin{cases} \tilde{\lambda}(\gamma(m,n)) := (\lambda|\gamma)\ e^{n\delta} \\ \tilde{\lambda}(z_{(1,0)}(m,n)) := e^{n\delta} \\ \tilde{\lambda}(z_{(0,1)}(0,0)) := \tilde{N} \\ \tilde{\lambda}(z_{(1,0)}(0,0)) := 1 \\ \tilde{\lambda}(d_0) := \dfrac{(\lambda|\lambda)}{2} \\ \tilde{\lambda}(\mathfrak{z}) := \ell+2 \end{cases}$$

Let $f : \mathbb{C}[\mathbb{Z}\delta] \longrightarrow H(\lambda)$ be the unique $\mathbb{C}$-linear map determined by $f(e^{p\delta}) := e^{\lambda+p\delta}\otimes 1$. We claim that f is a $\mathcal{B}$-module map. For $m,n\in\mathbb{Z}$ with $m+n\tilde{N}=0$, ie. if $\gamma(m,n) \in \tilde{\mathcal{H}}_0$ then

$$\begin{aligned} f(\gamma(m,n)\cdot e^{p\delta}) &= f\left((\lambda|\gamma)e^{n\delta}\cdot e^{p\delta}\right) \\ &= (\lambda|\gamma)\ f\left(e^{(n+p)\delta}\right) \\ &= (\lambda|\gamma)\ e^{\lambda+(n+p)\delta}\otimes 1. \end{aligned}$$

On the other hand,

$$\begin{aligned} \gamma(m,n)\cdot f(e^{p\delta}) &= T_m^{\gamma}(n\delta)(e^{\lambda+p\delta}\otimes 1) \\ &= \sum_{\ell\in\mathbb{Z}} \gamma(\ell) X_{-\ell+m}(n\delta)\left(e^{\lambda+p\delta}\otimes 1\right) \\ &= \sum_{m+nN\le\ell\le 0} \gamma(\ell)\cdot\left(e^{\lambda+(n+p)\delta}\otimes S_{\ell-(m+n\delta)}(n\delta)\right) \\ &= \gamma(0)\cdot(e^{\lambda+(n+p)\delta}\otimes 1), \text{ since } m+n\tilde{N}=0 \\ &= (\lambda|\gamma)\ e^{\lambda+(n+p)\delta}\otimes 1. \end{aligned}$$

The remaining calculations necessary to check that f is a $\mathcal{B}$-module map are immediate from earlier work, c.f. (4.2.5), (4.3.3), (1.4.4) and (3.3.1). Now by the universal property of the induced module, there is a unique $\tilde{\mathcal{H}}$-module map $\phi : M(\tilde{\lambda},\tilde{N}) \longrightarrow H(\lambda)$ such that $\phi \circ \iota = f$ where $\iota : \mathbb{C}[Z\delta] \longrightarrow M(\tilde{\lambda},\tilde{N})$ is the natural inclusion.

Proposition 4.6.2 If $\lambda \in \Gamma \backslash Q$, $\tilde{N} = (\lambda|\delta) \neq 0$ and $\tilde{\lambda}$ is given by (4.6.2) then as $\tilde{\mathcal{H}}$-modules we have $M(\tilde{\lambda},\tilde{N})/\ker\phi \cong H(\lambda)$.

Proof: This follows immediately from proposition 4.4.1. ∎

VERTEX REPRESENTATIONS OF THE VIRASORO-TORIODAL ALGEBRA

Introduction: As mentioned in the main introduction, the Fock space $\mathbb{C}[\Gamma]\otimes_{\mathbb{C}}S(\mathfrak{a}(Q)_-)$ is a completely decomposable module over the toroidal Lie algebra $\mathfrak{t}$. That is, in [MEY] it is proved that this space decomposes into a direct sum of indecomposable $\mathfrak{t}$-submodules.

In this chapter we prove our main result. Namely, with respect to the Virasoro-toroidal algebra, $\tilde{\mathfrak{t}} = \mathcal{V}ir \propto \mathfrak{t}$, we have the submodule decomposition $V(\Gamma) = \coprod_{m\in\mathbb{Z}} K(m)$ and $K(m)$ is an irreducible $\tilde{\mathfrak{t}}$-module if $m\neq 0$. $K(0)$ is not irreducible and we produce a filtration of submodules of it. Thus we will show that while the Fock space used in [MEY] is completely decomposable over $\mathfrak{t}$, our Fock space is "essentially" completely reducible over $\tilde{\mathfrak{t}}$.

§5.1 The Virasoro-Toroidal Lie algebra

Let $\tilde{\mathfrak{t}}$ be the Lie algebra of operators on $V(\Gamma) = \mathbb{C}[\Gamma]\otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma)_-)$ generated by $X_m(\alpha)$, L_k where $m,k \in \mathbb{Z}$ and $\alpha \in \Delta^{re} \subset Q$.

Proposition 5.1.1 $\tilde{\mathfrak{t}}$ is a semi-direct product of $\mathcal{V}ir$ and $\mathfrak{t}$.

Proof: A Lie algebra is a semi-direct product if it is a direct sum of a subalgebra and an ideal. We will show that $\mathfrak{t}$ is an ideal of $\tilde{\mathfrak{t}}$. It suffices to show that $[L_k, X_m(\alpha)] \in \mathfrak{t}$ where $k,m \in \mathbb{Z}$ and $\alpha \in \Delta^{re} \subset Q$. Indeed, by **CR4**

$$\sum_{m\in\mathbb{Z}} [L_k, X_m(\alpha)]z^{-m} = [L_k, X(\alpha,z)]$$

$$= z^k \left\{ \frac{k}{2}(\alpha|\alpha) + z\frac{d}{dz} \right\} X(\alpha,z)$$

$$= \sum_{n\in\mathbb{Z}} z^k \left\{ \frac{k}{2}(\alpha|\alpha) + z\frac{d}{dz} \right\} X_n(\alpha)z^{-n}$$

$$= \sum_{n\in\mathbb{Z}} \left(\frac{k}{2}(\alpha|\alpha) - n \right) X_n(\alpha)z^{-n+k}.$$

Now replacing n by m+k we have

$$[L_k, X_m(\alpha)] = \left\{ \frac{k}{2}(\alpha|\alpha) - (m+k) \right\} X_{m+k}(\alpha). \tag{5.1.1}$$ ∎

Note that $V(\Gamma) = \mathbb{C}[\Gamma]\otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma)_-) = \coprod_{m\in\mathbb{Z}} \left(\mathbb{C}[m\mu+Q]\otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma)_-) \right) =: \coprod_{m\in\mathbb{Z}} K(m)$.

It is clear from (4.3.4) that each $K(m)$, $m\in\mathbb{Z}$ is a $\tilde{\mathfrak{t}}$-module. In the next section we will study the structure of the $K(m)$, $m\in\mathbb{Z}$.

§5.2 The Structure of K(m), m≠0

We first consider the $\tilde{\mathfrak{t}}$-module K(m), m≠0. Thus $m\mu+Q \subset \Gamma\backslash Q$. We will need two formulas the first of which is a special case of the second and the second is obtained in exactly the same way as equation (4.3.3) in chapter 4:

(5.2.1) $X_{-(\gamma\,|\,\lambda+\frac{\gamma}{2})}(\gamma)\ (e^{\lambda}\otimes 1) = \varepsilon(\gamma,\lambda)\ (e^{\lambda+\gamma}\otimes 1)$, $\gamma\in Q$ and

(5.2.2) $X_r(\phi)\ (e^{\lambda}\otimes 1) = \varepsilon(\phi,\lambda)\ e^{\lambda+\phi}\otimes S_{-r-(\phi\,|\,\lambda+\frac{\phi}{2})}(\phi)$, $\phi\in Q$, $r\in\mathbb{Z}$.

Proposition 5.2.1: For m≠0, K(m) is an irreducible $\tilde{\mathfrak{t}}$-module.

Proof: It suffices to show

(b) $K(m) = \mathfrak{U}(\tilde{\mathfrak{t}})\cdot(e^{m\mu}\otimes 1)$ and

(a) every nonzero submodule N of K(m) contains $e^{m\mu}\otimes 1$.

For (a), we first note that

$K(m) = \coprod_{\alpha\in Q}\left(\mathbb{C}e^{m\mu+\alpha}\otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma)_-)\right) = \coprod_{\alpha\in Q} V_{\Gamma}(m\mu+\alpha)$. By (5.2.1), $\mathfrak{U}(\tilde{\mathfrak{t}})\cdot(e^{m\mu}\otimes 1)$ contains $e^{m\mu+\alpha}\otimes 1$ for every $\alpha\in Q$ and since $m\mu+\alpha \in \Gamma\backslash Q$ (m≠0) proposition 3.3.4 implies $\mathfrak{U}(\tilde{\mathfrak{t}})\cdot(e^{\lambda}\otimes 1) \supset \mathbb{C}e^{m\mu+\alpha}\otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma)_-)$, $\forall\alpha\in Q$. This establishes (a). We remark that the proof of proposition 3.3.4 relies, in general, on the use of all the Virasoro operators L_n, n<0.

To prove (b), let $0 \neq x := \sum_{i=1}^{r}(e^{m\mu+\alpha_i}\otimes f_i) \in N$, where $\alpha_i\in Q$, $f_i\in S(\mathfrak{a}(\Gamma)_-)$. We begin by establishing the following.

Claim: $\sum_{i=1}^{s} b_i(e^{m\mu+\beta_i}\otimes 1) \in N$, $b_i \in \mathbb{C}^\times$, $\beta_i \in Q$.

Indeed, this can be done by first successively acting on x with certain $\delta(n)$, $n>0$ thus "differentiating out" all indeterminates $\mu(-n)$, $n>0$. Similarly (by the proof of proposition 1.4.2 (ii) $\Rightarrow$ (iii)) we can differentiate out all indeterminates $\alpha_i(-n)$, $1\le i\le \ell$, $n>0$. Finally, acting with the Virasoro operators L_n, $n>0$, we can (using lemma 3.3.2) reduce x to an element $0 \neq y := \sum_{i=1}^{s} b_i(e^{m\mu+\beta_i}\otimes 1) \in N$, $b_i \in \mathbb{C}^\times$ and $\beta_i \in Q$. This establishes the claim.

In order to prove $(e^{m\mu}\otimes 1) \in N$, it suffices to show that (when $s>1$) we can reduce the length of y by acting on it with elements of $\tilde{\mathfrak{t}}$ and produce a nonzero element $z \in N$ with $1 \le \text{length}(z) < s$. For then by induction we can reduce to the case $e^{m\mu+\beta}\otimes 1 \in N$ and then the result follows from (5.2.1) by acting on this element with the operator $X_{(\beta\,|\,m\mu+\frac{\beta}{2})}(-\beta)$.

We now consider the following three mutually exclusive but exhaustive cases.

(i) For every $1\le i\le s$, $\beta_i = k_i\delta \in \mathbb{Z}\delta$

(ii) For every $1\le i\le s$, $\beta_i = \dot{\alpha} + k_i\delta$, $\dot{\alpha} \in \dot{Q}\setminus\{0\}$, $k_i \in \mathbb{Z}$

(iii) For every $1\le i\le s$, $\beta_i = \dot{\alpha}_i + k_i\delta$ where $k_i \in \mathbb{Z}$, $\dot{\alpha}_i \in \dot{Q}$ and for at least one pair $1\le p\neq q\le s$, $\dot{\alpha}_p \neq \dot{\alpha}_q$.

Case (i) is precisely the case covered in the proof of proposition 4.4.1 with $\lambda = m\mu$. We point out that the argument used there required the use of the Virasoro operator L_0. It is easy to see that one can reduce case (ii) to case (i). Indeed, by (5.2.1), we have

$$X_{(\dot{\alpha}\,|\,m\mu-\frac{\dot{\alpha}}{2})}(-\dot{\alpha})\ (y) = \sum_{i=1}^{s} \varepsilon_i b_i\left(e^{m\mu+k_i\delta}\otimes 1\right) \text{ where } \varepsilon_i = \varepsilon(m\mu+\beta_i,-\dot{\alpha}) \in \{\pm 1\}.$$

Finally, in order to handle case (iii), we introduce

$\dot{Q}^* := \{\dot{\alpha}+n\delta : \dot{\alpha} \in \dot{Q}\backslash\{0\},\ n\in\mathbb{Z}\} = Q\backslash\mathbb{Z}\delta$. Note that $\dot{Q}\backslash\{0\} \subset \dot{Q}^* \subset Q$ and that since $\dot{Q}$ is positive definite we have

(5.2.3) There does not exist a $\tau\in\dot{Q}^*$ such that $(\tau|\dot{Q}^*) \equiv 0$.

Next let $1\le p\ne q\le s$ be such that $\dot{\alpha}_p \ne \dot{\alpha}_q$. Then $\beta_p-\beta_q \in \dot{Q}^*$. Choose $\gamma \in \dot{Q}^*$ such that $(\gamma|\beta_p) \ne (\gamma|\beta_q)$. Such a γ exists since otherwise $(\beta_p-\beta_q|\dot{Q}^*) \equiv 0$ contrary to (5.2.3).

Now consider the set $\{(\beta_i|\gamma) : 1\le i\le s\} \subset \mathbb{C}$ which ,from what we just saw, contains at least two distinct elements. Let $\beta \in \{\beta_1,\dots,\beta_s\}$ be an element satisfying $(\gamma|\beta) \le (\gamma|\beta_i)$, $1\le i\le s$. Of course β may not be unique but it obviously exists. Relabelling if necessary, we can assume without loss of generality that $\beta=\beta_1$. Thus we have

(5.2.4)
$$\begin{cases} (\gamma|\beta_1) \le (\gamma|\beta_i), \text{ for all } 2\le i\le s \\ (\gamma|\beta_1) < (\gamma|\beta_i), \text{ for some } 2\le i\le s. \end{cases}$$

Thus by (5.2.1) and (5.2.2) we have

$$z := X_{-(\gamma|m\mu+\beta_1+\frac{\gamma}{2})}(\gamma)\ (y)$$

$$= \varepsilon_1 b_1 (e^{m\mu+\beta_1+\gamma}\otimes 1) + \sum_{i=2}^{s} \varepsilon_i b_i \left(e^{m\mu+\beta_1+\gamma}\otimes S_{(\gamma|m\mu+\beta_1+\frac{\gamma}{2})-(\gamma|m\mu+\beta_1+\frac{\gamma}{2})}(\gamma)\right)$$

$$= \underbrace{\varepsilon_1 b_1 (e^{m\mu+\beta_1+\gamma}\otimes 1)}_{\ne 0} + \sum_{i=2}^{s} b_i \varepsilon_i \left(e^{m\mu+\beta_1+\gamma}\otimes S_{(\gamma|\beta_1)-(\gamma|\beta_1)}(\gamma)\right)$$

Now since $(\gamma|\beta_1)-(\gamma|\beta_i)\le 0$, $2\le i\le s$ (5.2.4) gives $1 \le \text{length}(z) < s$ as required.■

§5.3 The Structure of K(0)

We end this chapter by pointing out that the filtration of $\mathcal{V}ir$-submodules $W^k(n\delta)$ of $W(n\delta)$ from chapter 2 can be used to produce a filtration of $\tilde{\mathfrak{t}}$-submodules of K(0).

For $\ell\in\mathbb{Z}$ define the following $\mathbb{C}$-subspaces of K(0):

$$K^{\ell}(0) := \mathbb{C}[Q]\otimes_{\mathbb{C}} S(\mathfrak{a}(\dot{Q})_-)\Theta^{\ell} = \coprod_{\alpha\in Q} V^{\ell}(\alpha). \tag{5.3.1}$$

Proposition 5.3.1: For $\ell\in\mathbb{Z}$, $K^{\ell}(0)$ is a $\tilde{\mathfrak{t}}$-submodule of K(0).

Proof: This follows from (4.3.4) and the fact that each $V^{\ell}(\alpha)$ is an $\tilde{\mathfrak{a}}$-module. ∎

For each $\ell\in\mathbb{Z}$, define $K_{\ell}(0) := K^{\ell}(0)/K^{\ell-1}(0)$. As in §4.5 one can show that $K_{\ell}(0) = \coprod_{\gamma\in I_{\ell}} K_{\ell}^{\gamma}(0)$ where $K_{\ell}^{\gamma}(0)$ is defined as the cyclic $\tilde{\mathfrak{t}}$-submodule of $K_{\ell}(0)$ generated by $\overline{e^0\otimes\xi_{\ell}^{\gamma}}$ which, as a $\mathbb{C}$-space, is isomorphic to $\coprod_{\alpha\in Q} V_{\ell}^{\gamma}(\alpha)$.

APPENDIX A

Geometric Lattices

A *geometic lattice* is a pair $(L,(\cdot|\cdot))$ consisting of a free $\mathbb{Z}$-module L of finite rank together with a symmetric bilinear form $(\cdot|\cdot) : L \times L \longrightarrow \mathbb{Z}$. Let $\ell := \mathbb{C}\otimes_{\mathbb{Z}} L$ denote the complexification of L. Two lattices $(L,(\cdot|\cdot))$ and $(L',(\cdot|\cdot)')$ are *isomorphic* if there is a group isomorphism $f : L \longrightarrow L'$ satisfying $(x|y) = (f(x)|f(y))'$, $x,y \in L$. One can associate various *invariants* to a geometric lattice in order to distinguish it from others (up to isomorphism). A few examples are rank, determinant, signature and type.

The *rank* simply refers to the rank of L as a free $\mathbb{Z}$-module. The *determinant* of a lattice is the determinant of the matrix of the bilinear form $(\cdot|\cdot)$ relative to any $\mathbb{Z}$-basis. A geometric lattice is said to be *nondegenerate* or *nonsingular* if its determinant is nonzero. By Sylvester's theorem, we can write ℓ as $\ell_+\oplus\ell_-$ where ℓ_+ (resp. ℓ_-) is positive definite (resp. negative definite). The dimensions m_+ and m_- of the spaces ℓ_+ and ℓ_- are invariants and hence so also is the *signature* $\mathrm{sgn}(L) := m_+ - m_-$. If $\mathrm{sgn}(L) = \mathrm{rank}L$ we say that L is *positive definite*. Finally, if $(x|x) \in 2\mathbb{Z}$ for all $x\in L$ then we say that L is *even*. Otherwise we say L is *odd*.

A *sublattice* of a geometric lattice L is defined in the obvious way. We say that L' is a *full sublattice* of L if $\mathrm{rank}L = \mathrm{rank}L'$. Suppose L' is a full sublattice of L. The *index* of L' in L, denoted [L:L'], is the number of left cosets of L' in L and by the structure theorem for finitely generated modules over a PID (such as $\mathbb{Z}$) we know that [L:L'] is finite. In fact, if $\{e_i\}_{i=1}^n$ is a basis for L and $\{f_i\}_{i=1}^n$ a basis for L', with the bases related by some matrix A, then one can show that $[L:L'] = |\det A|$ and that $\det L' = [L:L']^2 \det L$.

Before we turn to the main examples, we make one more observation. If L is a geometric lattice and $L_0 := \{x \in L : (x|x)=0 \bmod 2\}$ then L_0 is an even sublattice of L. Since $2L = \{2x : x \in L\} \subset L_0$ it is clear that L_0 is a full sublattice of L. Note that if L is even then $L_0 = L$. Otherwise, one can easily show that $[L:L_0] = 2$ and hence, by what we have just seen, $\det L_0 = 4 \det L$.

Example (Type D_ℓ lattices)

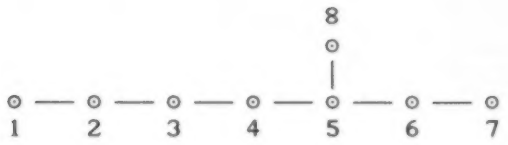
Consider the $\ell \times \ell$ identity matrix I_ℓ^+. Define $D_\ell := \{ x \in I_\ell^+ : (x|x) = 0 \bmod 2\}$. Three things can be immediately observed: (i) D_ℓ is positive definite since $D_\ell \subset I_\ell^+$, (ii) D_ℓ is even since $x \in D_\ell \Leftrightarrow (x|x) = 0 \bmod 2$ and (iii) $\det D_\ell = 4$. Next, if $\varepsilon_1,\ldots,\varepsilon_\ell$ is the standard orthonormal basis of I_ℓ^+ and $\ell \geq 2$ then the ℓ elements $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$, $1 \leq i \leq \ell-1$ and $\alpha_\ell := \varepsilon_{\ell-1} + \varepsilon_\ell$ form a basis for D_ℓ. Note that each α_i, $1 \leq i \leq \ell$ has norm 2 and $(\alpha_i|\alpha_j) = 0$ if $|i-j| \geq 2$ and $(\alpha_{\ell-1}|\alpha_\ell) = 0$. Thus the matrix of D_ℓ can be described as follows. It has a diagonal of 2×2 matrices and zeroes elsewhere. Each of the 2×2 matrices starting from the top left is $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ except the last one which is $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

Example (Type A_ℓ lattices)

Consider the $(\ell+1)\times(\ell+1)$ identity matrix $I_{\ell+1}^+$ with standard basis $\varepsilon_1,\ldots,\varepsilon_{\ell+1}$. Define $A_\ell := \left\{ \sum_{i=1}^{\ell+1} n_i \varepsilon_i : \sum_{i=1}^{\ell+1} n_i = 0 \right\}$. It is easy to check that A_ℓ is an even positive definite lattice with basis $\{\alpha_i := \varepsilon_i - \varepsilon_{i+1}, 1 \leq i \leq \ell\}$ and hence $\det(A_\ell) = 2\det(A_{\ell-1}) - \det(A_{\ell-2})$ with $\det A_1 = 2$, $\det A_2 = 3$. Thus $\det(A_\ell) = \ell+1$.

Example (Type E_ℓ lattices)

Fix $\ell \in \{6,7,8\}$ and let $E_\ell := \coprod_{i=1}^{\ell} \mathbb{Z}\alpha_i$. Consider the diagram



Define $(\cdot|\cdot) : E_\ell \times E_\ell \longrightarrow \mathbb{Z}$ by

$$\begin{cases} (\alpha_i|\alpha_i) := 2, \ 1 \le i \le \ell \\ (\alpha_i|\alpha_j) := -1, \text{ if } i \ne j \text{ and there is an edge joining nodes i and j} \\ (\alpha_i|\alpha_j) := 0, \text{ otherwise.} \end{cases}$$

One can show that each of E_6, E_7, and E_8 are even, positive definite geometric lattices with determinants 1, 2 and 3 respectively. Proofs of these facts can be found in chapter 3 of [MP-2].

APPENDIX B

Universal Covering Algebras

Definitions Let $\mathfrak{g}$ be a Lie algebra. By a *central extension* of $\mathfrak{g}$ we mean an exact sequence of Lie algebras

(CE) $$0 \longrightarrow \mathfrak{c} \xrightarrow{\iota} \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

such that $\iota(\mathfrak{c})$ is contained in the centre of $\mathfrak{e}$.

Given another central extension

(CE)′ $$0 \longrightarrow \mathfrak{c}' \longrightarrow \mathfrak{e}' \longrightarrow \mathfrak{g} \longrightarrow 0$$

of $\mathfrak{g}$, by a *morphism* from (CE) to (CE)′ we mean a pair (ϕ,ϕ') of Lie algebra homomorphisms such that the following diagram commutes:

$$\begin{array}{ccccc} \mathfrak{c} & \longrightarrow & \mathfrak{e} & \longrightarrow & \mathfrak{g} \\ \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{id.}} \\ \mathfrak{c}' & \longrightarrow & \mathfrak{e}' & \longrightarrow & \mathfrak{g}' \end{array}$$

If $\mathfrak{e}$ is perfect, i.e. if $[\mathfrak{e},\mathfrak{e}] = \mathfrak{e}$, then we call (CE) a *covering* of $\mathfrak{g}$. One can then establish

Proposition If a central extension of a Lie algebra $\mathfrak{g}$ is a covering then there exists at most one morphism to any other central extension of $\mathfrak{g}$.

Definition A covering of a Lie algebra $\mathfrak{g}$ is said to be *universal* if there exists a unique morphism from it to any other central extension of $\mathfrak{g}$.

Consider again the central extension (CE). Choose a subspace $\mathfrak{g}'$ of $\mathfrak{e}$ isomorphic to $\mathfrak{g}$ under π. $\mathfrak{g}'$ can easily be constructed as follows. Let $\{x_j\}_{j\in J}$ be a basis of $\mathfrak{g}$ and let $x_j' \in \mathfrak{e}$ be chosen so that $\pi(x_j') = x_j$. Then the $\mathbb{C}$-space spanned by the x_j' is such a $\mathfrak{g}'$. Let $' : \mathfrak{g} \longrightarrow \mathfrak{g}'$ be the inverse map of π. Then $\pi([x,y]'-[x',y']) = 0$ and hence $\alpha(x,y) := [x,y]'-[x',y'] \in \mathfrak{c}$ is a bilinear map satisfying

$$\textbf{(CC)} \qquad \begin{cases} \text{(i) } \alpha(x,y) = -\alpha(y,x) \\ \text{(ii) } \alpha(x,[y,z]) + \alpha(y,[z,x]) + \alpha(z,[x,y]) = 0 \end{cases}$$

We call α a *cocycle* on $\mathfrak{g}$ with coefficients in $\mathfrak{c}$.

Conversely, given a cocycle α on $\mathfrak{g}$ with coefficients in $\mathfrak{c}$ we can construct a central extension of $\mathfrak{g}$ as follows. Let $\mathfrak{e} := \mathfrak{g}\oplus\mathfrak{c}$ as a $\mathbb{C}$-space. Define a multiplication $[\cdot,\cdot]_*$ on $\mathfrak{e}$ by $[x+a,y+b]_* = [x,y] + \alpha(x,y)$, $x,y\in\mathfrak{g}$ and $a,b\in\mathfrak{c}$. Properties **(CC)**(i) and (ii) guarantee that $\mathfrak{e}$ is a Lie algebra. The exact sequence $0 \longrightarrow \mathfrak{c} \xrightarrow{\iota} \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$ where ι and π are the canonical projections and injections respectively is a central extension of $\mathfrak{g}$.

Finally, we turn our attention to the question of which Lie algebras admit universal coverings.

Proposition For a Lie algebra to admit a universal central extension it is necessary and sufficient that $\mathfrak{g}$ be perfect.

Sketch of proof: If $\mathfrak{g}$ admits a universal covering of the form (CE), then $\mathfrak{e}$ is perfect and $\pi(\mathfrak{e})=\mathfrak{g}$. However, π is a homomorphism so $\mathfrak{g}$ is also perfect.

Conversely, let $\Lambda^2(\mathfrak{g}) := T^2(\mathfrak{g})/J$ where J is the subspace of $T^2(\mathfrak{g}) = \mathfrak{g}\otimes\mathfrak{g}$ spanned by all elements of the form $x\otimes y - y\otimes x$, $x,y\in\mathfrak{g}$. Let $x\wedge y$ denote the image of $x\otimes y$ in $\Lambda^2(\mathfrak{g})$. Then $x\wedge x = 0$ in $\Lambda^2(\mathfrak{g})$. Let $V := \Lambda^2(\mathfrak{g})/I$ where I is the subspace spanned by all elements of the form $x\wedge[y,z] + y\wedge[z,x] + z\wedge[x,y]$, $x,y,z \in \mathfrak{g}$. Let $\alpha(x,y)$ denote the image of $x\wedge y$ in $\Lambda^2(\mathfrak{g})/I$. Define $\mathfrak{e} := \mathfrak{g}\oplus V$ and a multiplication $[\cdot,\cdot]_{\mathfrak{e}}$ on $\mathfrak{e}$ by $[x+v,y+w] := [x,y] + \alpha(x,y)$. Then $\mathfrak{e}$ is a Lie algebra. However, the central extension $0 \longrightarrow V \xrightarrow{\iota} \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$ is not necessarily even a covering since $\mathfrak{e}$ need not be perfect. One remidies this by defining $\hat{\mathfrak{e}} := [\mathfrak{e},\mathfrak{e}]_{\mathfrak{e}}$ and $\hat{V} := \mathfrak{e} \cap V$. Then one can show that $\hat{\mathfrak{e}}$ is perfect and that $0 \longrightarrow \hat{V} \longrightarrow \hat{\mathfrak{e}} \longrightarrow \mathfrak{g} \longrightarrow 0$ is the universal covering of $\mathfrak{g}$. ■

Example: The Virasoro algebra with generators $\{d_k, \mathfrak{z} : k\in\mathbb{Z}\}$ and relations $[d_k,d_\ell] = (k-\ell)d_{k+\ell} + \frac{1}{12}\,\delta_{k+\ell,0}(k^3-k)\mathfrak{z}$, $k,\ell\in\mathbb{Z}$ and $\mathfrak{z}$ is central is the universal covering algebra of the *Witt algebra* with generators d_k, $k\in\mathbb{Z}$ and relations $[d_k,d_\ell] = (k-\ell)\,d_{k+\ell}$, $k,\ell\in\mathbb{Z}$.

For more details on the results of this section, consult [MP-2]. A standard reference for central extensions is [Ga].

APPENDIX C

Central extensions of lattices and twisted group algebras

Let L be an abelian group. By a *central extension* $(\pi,\hat{L})$ of L by an abelian group A we mean an exact sequence of groups

$$0 \longrightarrow A \xrightarrow{\iota} \hat{L} \xrightarrow{\pi} L \longrightarrow 0$$

where $\iota(A)$ is in the centre of $\hat{L}$. We use additive notation for L and multiplicative notation for $\hat{L}$.

Let $\phi : L \longrightarrow \hat{L}$ be a *section* of L, i.e. a map satisfying $\pi\circ\phi = id_L$. Then every element of $\hat{L}$ is uniquely expressible as $\phi(\alpha)\iota(a)$, $\alpha\in L$ and $a\in A$. Indeed, if $x \in \hat{L}$ let $\alpha := \pi(x)$ and consider $\phi(\alpha)^{-1}x$. Since $\pi(\phi(\alpha)^{-1}x) = \alpha\alpha^{-1} = 1$, $\phi(\alpha)^{-1}x \in \ker\pi \in \iota(A)$ so $x = \phi(\alpha)\iota(a)$ for some $a\in A$. We will denote the element $\phi(\alpha)\iota(a)$ simply by (α,a).

Let $\varepsilon : L \times L \longrightarrow A$ be defined by $\varepsilon(\alpha,\beta) := \iota^{-1}\left(\phi(\alpha+\beta)^{-1}\phi(\alpha)\phi(\beta)\right)$, which makes sense since ι is injective and $\phi(\alpha+\beta)^{-1}\phi(\alpha)\phi(\beta) \in \ker\pi = \iota(A)$. Then one shows that $(\alpha,a)(\beta,b) = (\alpha+\beta,\varepsilon(\alpha,\beta)+a+b)$ and hence ε allows us to express the structure of $\hat{L}$ in terms of the structure of A and L. Note that by associativity we have

(CC) $$\varepsilon(\alpha+\beta,\gamma)\ \varepsilon(\alpha,\beta) = \varepsilon(\alpha,\beta+\gamma)\ \varepsilon(\beta,\gamma),\ \alpha,\beta,\gamma \in L.$$

We call a map $\varepsilon : L \times L \longrightarrow A$ satisfying conditions **(CC)** a *2-cocycle of L with coefficients in A*. A simple way to get a 2-cocycle is to take any bimultiplicative map $\varepsilon : L \times L \longrightarrow A$. In this case, **(CC)** holds trivially.

We now restrict our attention to the case where A is the field with two elements $\mathbb{F}_2 = \{\pm 1\}$. Let $2L = \{2\alpha : \alpha\in L\}$ and observe that $\overline{L} := L/2L$ is a vector space over $\mathbb{F}_2$. Since $\varepsilon(2\alpha,\beta) = 1 = \varepsilon(\alpha,2\beta)$, $\alpha,\beta\in L$ any bimultiplicative map $\varepsilon : L \times L \longrightarrow \mathbb{F}_2$ induces a bilinear form $\overline{\varepsilon} : \overline{L} \times \overline{L} \longrightarrow \mathbb{F}_2$.

Let V be any vector space over $\mathbb{F}_2$. A mapping $q : V \longrightarrow \mathbb{F}_2$ is called a *quadratic form on V* if the associated mapping

(QF) $$b(x,y) := q(x+y) - q(x) - q(y)$$

is bilinear. Let Q(V) be the vector space of all quadratic forms on V over $\mathbb{F}_2$; B(V) the bilinear forms on V and $B_0(V) := \{b \in B(V) : b(x,x) = 0 \ \forall x \in V\}$.

Now given a bilinear form ε on V we may define a quadratic form q_ε on V by $q_\varepsilon(x) := \varepsilon(x,x)$. Thus we have a map $B(V) \longrightarrow Q(V) : \varepsilon \longmapsto q_\varepsilon$ with kernel $B_0(V)$. By **(QF)** the bilinear form associated to q_ε is given by

(*) $$b_\varepsilon(x,y) = q_\varepsilon(x+y) - q_\varepsilon(x) - q_\varepsilon(y) = \varepsilon(x,y) + \varepsilon(y,x).$$

In fact, one can establish that

(**) $$0 \longrightarrow B_0(V) \longrightarrow B(V) \longrightarrow Q(V) \longrightarrow 0$$

is an exact sequence. With this result we now turn to the problem of finding a cocycle for an even geometric lattice.

Proposition: Let L be an even geometric lattice. Then there is a central extension $\hat{L}$ of L by $\{\pm 1\}$ with 2-cocycle $\varepsilon : L \times L \longrightarrow \{\pm 1\}$ satisfying

$$(1)\ \varepsilon(\alpha,\alpha) = (-1)^{(\alpha|\alpha)/2}$$

$$(2)\ \varepsilon(\alpha,\beta)\ \varepsilon(\beta,\alpha) = (-1)^{(\alpha|\beta)}.$$

Proof: Let $\overline{L} := L/2L$, a vector space over $\mathbb{F}_2$. Consider the quadratic form $q(\overline{\alpha}) := (-1)^{(\overline{\alpha}|\overline{\alpha})/2}$ with associated bilinear form $b(\overline{\alpha},\overline{\beta}) = (-1)^{(\overline{\alpha}|\overline{\beta})}$. Applying the exactness of sequence **(**)** we have a bilinear form $\overline{\varepsilon} : \overline{L} \times \overline{L} \longrightarrow \mathbb{F}_2$ such that $q(\overline{\alpha}) = \overline{\varepsilon}(\overline{\alpha},\overline{\alpha})$ and by **(*)** we have

$(1)\ \overline{\varepsilon}(\overline{\alpha},\overline{\beta})\ \overline{\varepsilon}(\overline{\beta},\overline{\alpha}) = b(\overline{\alpha},\overline{\beta}) = (-1)^{(\alpha|\beta)}$ and

$(2)\ \overline{\varepsilon}(\overline{\alpha},\overline{\alpha}) = (-1)^{(\overline{\alpha}|\overline{\alpha})/2}$.

Finally, one simply defines $\varepsilon : L \times L \longrightarrow \mathbb{F}_2$ by $\varepsilon(\alpha,\beta) := \overline{\varepsilon}(\overline{\alpha},\overline{\beta})$. ∎

We now turn to the construction of the twisted group algebra of a geometric lattice from a central extension $\hat{L}$ of L. Begin with a central extension $0 \longrightarrow \{\pm 1\} \xrightarrow{\iota} \hat{L} \xrightarrow{\pi} L \longrightarrow 0$ with section ϕ. Let $z := \iota(1)$ so that $z^2=1$. We will use the notation e^{α} for $\phi(\alpha)$, $\alpha \in L$. As before, any element in $\hat{L}$ may be written in the form $e^{\alpha}z^n$, $\alpha \in A$, $n \in \{\pm 1\}$ and multiplication in $\hat{L}$ is given by

$$(***) \qquad (e^{\alpha}z^n)(e^{\beta}z^m) = e^{\alpha+\beta}z^{\varepsilon(\alpha,\beta)+n+m}.$$

Let $\mathbb{C}[\hat{L}]$ be the group algebra over $\hat{L}$. Then $\mathbb{C}[\hat{L}]$ has a $\mathbb{C}$-basis $\left\{e^{\alpha}z^n : \alpha \in L,\ n \in \{\pm 1\}\right\}$ and multiplication is given by bilinear extension of (***). Next one shows that $j = \coprod_{\alpha \in L} \mathbb{C}(e^{\alpha}+e^{\alpha}z)$ is a two-sided ideal in $\mathbb{C}[\hat{L}]$ generated by the central element $1+z$. Let $\mathbb{C}[L]$ be the group algebra of L with basis $\{e(\alpha) : \alpha \in L\}$ and multiplication given by bilinear extension of $e(\alpha)\, e(\beta) = e(\alpha+\beta)$. Note that the linear map $f : \mathbb{C}[\hat{L}] \longrightarrow \mathbb{C}[L]$ given by $e^{\alpha}z^n \longrightarrow (-1)^n \cdot e(\alpha)$ is surjective with kernel precisely j. Thus, as $\mathbb{C}$-spaces, we have $\mathbb{C}[\hat{L}]/j \cong \mathbb{C}[L]$. Now consider the $\mathbb{C}$-space $\mathbb{C}[L]$ with new multiplication, namely the one induced by this isomorphsim. One easily checks that this multiplication is given by

$$e^{\alpha}e^{\beta} = (-1)^{\varepsilon(\alpha,\beta)}e^{\alpha+\beta}.$$

The resulting algebra is called *the twisted group algebra associated to L.*

GLOSSARY

Symbol	Definition and/or Name	Page
$\mathcal{H}(L,n)$	Generalized Heisenberg algebra (GHA)	24-25
$\coprod$	direct product of several $\mathbb{C}$-spaces	24
$\oplus$	direct sum of two $\mathbb{C}$-spaces	24
$\mathbb{C}$	complex numbers	24
$\mathbb{Z}$	integers	24
$\mathfrak{z}$	central part of GHA	24
$\mathfrak{a}(L)$	notation for $\mathcal{H}(L,1)$	27
Q	$= \dot{Q} \oplus \mathbb{Z}\delta$ (positive semi-definite lattice)	28
Ω_A/dA	module of differentials	29-30
$\dot{\mathfrak{g}}$	simple finite dimensional Lie algebra over $\mathbb{C}$	29
$\tau_{[n]}$	toroidal Lie algebra	31
$\mathcal{H}$	$= \mathcal{H}(\dot{Q},2)$	34
$\mathfrak{L}$	subalgebra of $\mathcal{H}$ isomorphic to $\mathfrak{a}(Q)$	35
$S(\mathfrak{a}(L)_-)$	Fock space corresponding to L	37
$V_L(\lambda)$	canonical representation of $\mathfrak{a}(L)$	38-39
Γ	$= \dot{Q} \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\mu$	42
$M(\alpha)$	Induced (Verma) module for $\mathfrak{a}(L)$	43
$\mathfrak{U}(\mathfrak{a}(L))$	universal enveloping algebra of $\mathfrak{a}(L)$	43
$(\cdot \mid \cdot)$	Shapavalov form	45
Λ	$= \mathbb{Z}\delta \oplus \mathbb{Z}\mu$ (Hyperbolic lattice)	45,55
$\mathcal{V}ir$	Virasoro algebra	49

Symbol	Definition and\or Name	Page
$\mathfrak{z}$	central element in $\mathcal{V}ir$	49
M(c,h)	Verma module for $\mathcal{V}ir$	49
L(c,h)	unique irreducible quotient of M(c,h)	49
:—:	normal ordering	50
L_k	Virasoro operators	50
$\{\alpha_1,\alpha_2\}$	orthonormal basis for $\mathfrak{p} = \mathbb{C}\otimes_{\mathbb{Z}}\Lambda$	55
Θ	symmetric algebra of $\coprod_{n>0}\mathfrak{p}(-n)$	55
D	subalgebra $S\left(\coprod_{n>0}\mathbb{C}\delta(-n)\right)$ of Θ	57
M	subalgebra $S\left(\coprod_{n>0}\mathbb{C}\mu(-n)\right)$ of Θ	57
$W(\lambda)$	notation for Fock space $V_\Lambda(\lambda)$	57
H_n	Virasoro operators on $W(\lambda)$	55
$H_n^{\pm}$	$= H_n^+ + H_n^-$	58
Ω_j	$\mathbb{C}$-span of monomials in M of length j	61
$\Omega_{\leq j}$	$\mathbb{C}$-span of monomials in M of length $\leq$j	61
Δ_j	$\mathbb{C}$-span of monomials in D of length j	61
Θ^k	$= \coprod_{j\geq 0}\left(\Omega_{\leq(j+k)}\Delta_j\right) \subset \Theta$	62
$W^k(\lambda)$	$= \mathbb{C}e^{\lambda}\otimes_{\mathbb{C}}\Theta^k \subset W(\lambda)$	62
$\overline{W_k(0)}$	$= \mathbb{C}e^{0}\otimes_{\mathbb{C}}\coprod_{j\geq 0}\left(\Omega_{(j+k)}\Delta_j\right) \subset W(0)$	61

Symbol	Definition and/or Name	Page
$W_k(\lambda)$	subquotient $W^k(\lambda)/W^{k-1}(\lambda)$	62
$(\cdot \mid \cdot)$	Hermitian form on Θ	66
$\mathfrak{S}_r$	symmetric group on r symbols	66
$c(n,m)$	$\text{card}\{\sigma \in \mathfrak{S}_r : \sigma(n) = m\}$	67
$c(n)$	$= c(n,n)$	67
$\delta_{n,m}$	Generalized kronecker delta	67
$\pi(n)$	$\prod_{i=1}^{r} n_i$	67
ϑ	involution on Θ	69
ω	involution on $W(\lambda)$	69
$<\cdot \mid \cdot>$	Hermitian form on $W(\lambda)$	69
$\overline{W}_k$	notation for $\overline{W_k(0)}$	77
$W_k(n)$	subspace of $W_k \equiv W_k(n)$ of elements of degree n	81
$p_r(s)$	partitions of r into s parts	81
$\tilde{\mathfrak{a}}(L)$	semi-direct product algebra $\mathcal{V}ir \propto \mathfrak{a}(L)$	89
$\tilde{\mathfrak{a}}$	$= \tilde{\mathfrak{a}}(Q)$	89
$V(\lambda)$	$= V_\Gamma(\lambda)$	91
$\hat{\mathfrak{a}}$	$= \mathcal{V}ir_+ \propto \mathfrak{a}$ thus $\mathfrak{a} \subset \hat{\mathfrak{a}} \subset \tilde{\mathfrak{a}}$	93
$\mathcal{P}_k$	set of partitions of k	94
d_m	$= d_{m_1} d_{m_2} \cdots d_{m_r} \in \mathfrak{U}(\mathcal{V}ir_+)$, $m = (m_1,\ldots,m_r) \in \mathbb{Z}_+^r$	95
—	overbar denoting omission	99

Symbol	Definition and/or Name	Page
$[\cdot]$	greatest integer less than	99
V	$= V_\Gamma(\lambda)$	102
ϕ	isomorphism $V_Q^*(\alpha)\otimes_{\mathbb{C}} W(\lambda) \cong V$	102
V^k	submodule of V for $k\in\mathbb{Z}$	103
I_k	indexing subset	103
V_k^γ	submodule of V_k	104
$\gamma(m,n)$	generator of $\mathcal{H}$	108
$z_{(m,n)}(r,s)$	central element of $\mathcal{H}$	108
$\tilde{\mathcal{H}}$	Virasoro-Heisenberg algebra	108
ε	cocycle on Q	110
$V(\Gamma)$	$= \mathbb{C}[\Gamma]\otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma_-))$, Full Fock space on Γ	110
$X(\alpha,z)$	vertex operator for α	111
$X_n(\alpha)$	moment of $X(\alpha,z)$	111
CR0-CR4	commutation relations of vertex operators	112
$\overline{S(\mathfrak{a}(\Gamma)_-)}$	formal completion of $S(\mathfrak{a}(\Gamma)_-)$	111
Δ, Δ^{re}	root system and real roots of affine algebra	112
$T_m^\gamma(n\delta)$	$= \sum_{k\in\mathbb{Z}} :\gamma(k)X_{-k+m}(n\delta):$	113
$H(\lambda)$	$\mathbb{C}[\lambda+\mathbb{Z}\delta]\otimes_{\mathbb{C}} S(\mathfrak{a}(\Gamma)_-)$ an $\tilde{\mathcal{H}}$-module	116
$S_r(\alpha)$	elementary Schur polynomial	116
$H^k(\lambda)$	$= \mathbb{C}[\lambda+\mathbb{Z}\delta]\otimes_{\mathbb{C}} S(\mathfrak{a}(\dot{Q})_-)\Theta^k \subset H(\lambda)$	122

Symbol	Definition and/or Name	Page
$H_k(\lambda)$	$= H^k(\lambda)/H^{k-1}(\lambda)$	122
$H_k^{\gamma}(\lambda)$	cyclic submodule of $H_k(\lambda)$ generated by $e^{\lambda}\otimes\xi_k^{\gamma}$	122
$\tilde{\mathcal{H}}_-\oplus\tilde{\mathcal{H}}_0\oplus\tilde{\mathcal{H}}_+$	decomposition of $\tilde{\mathcal{H}}$	124-125
$\mathfrak{B}$	$= \tilde{\mathcal{H}}_0\oplus\tilde{\mathcal{H}}_+$	125
$M(\tilde{\lambda},\tilde{N})$	Verma-type induced module	125
$\tilde{\mathfrak{t}}$	Virasoro-toroidal algebra	129
$K(m)$	$= \mathbb{C}[m\mu+Q]\otimes_{\mathbb{C}}S(a(\Gamma)_-)$ a $\tilde{\mathfrak{t}}$-module	129
$K^{\ell}(0)$	$= \mathbb{C}[Q]\otimes_{\mathbb{C}}S(a(\dot{Q})_-)\Theta^{\ell}$ a $\tilde{\mathfrak{t}}$-submodule of $K(0)$	133
$K_{\ell}(0)$	$= K^{\ell}(0)/K^{\ell-1}(0)$ subquotient of $K(0)$	133.

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